# Model Order Reduction vs. Structural Monitoring

F. CASCIATI, S. CASCIATI, L. FARAVELLI
and M. FRANCHINOTTI

## ABSTRACT

The goal of Model Order Reduction (MOR) is to catch a model of order lower than that of the real model satisfactory for the purpose of the analysis. The reduced order model should be characterized by a low computational effort but also to be able to estimate the input-output map of the original system in an important region of the input space. Actually, since only a subset of the input space is of interest, this matching should occur in this subset of the input space.

This contribution emphasizes some consequences of the adoption of a reduced order model when structural monitoring applications are pursued.

## INTRODUCTION

A Model Order Reduction (MOR) technique [1] was strictly required, in order to be able to achieve a solution of large-size dynamic structural problems, until one decade ago, due to the computation hardware capability available at that time. Nowadays, the MOR strategy, which offers a low computational effort coupled with the ability to estimate, in an important region of the input space, the input-output mapping of the original system, remains a viable methodology toward the implementation of real time control systems and/or the solution of problems where several repeated analyses must be carried out (e.g., optimization and/or reliability problems) .

Within the MOR techniques, the current research efforts [2] are mainly addressed to capture lower order models for nonlinear systems. Nevertheless, also the correct application of MOR when dealing with linear systems still requires attention. In this paper, a linear, time invariant structural problem is studied. The manner in which the reduction cascade can affect the accuracy of the final reduced model was discussed in two previous studies. In [3] one can see how two different cascades of model order reduction are applied to a benchmark building and appreciate the different accuracy in two different 20 states reduced order models. In [4], as well as in this paper, a smaller size frame is investigated.

F. Casciati, L.Faravelli, University of Pavia, via Ferrata 1, Pavia, Italy
S.Casciati, University of Catania, Italy

## CASE STUDY

To verify the feasibility of structural control schemes in Civil and Infrastructural Engineering suggested the introduction of benchmark studies. The building analyzed in one of them was first introduced in [5]: the original 20 story building is modified in this paper to take into account the 3 upper stories only. The resulting building is the 3 stories steel frame shown in Figure 1, with an inter-storey height of 3.96 m, for a total height of 11,88 m above the ground level, and 5 bays of 6.10 m each along the N-S direction. The steel yielding stress is 345 MPa. The masses to be considered are $5.32 \times 10^5$ kg for all the levels.

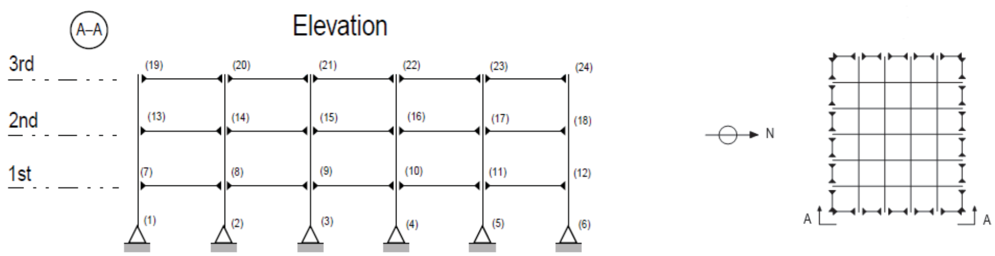


Figure 1: The case study investigated within this paper, re-elaborated from [4] and [5].

For the purposes of this paper, the frame was subjected to a single ground acceleration time history, i.e., the N-S component of the acceleration recorded during the El Centro seismic event of May 18, 1940. The peak acceleration is 3.417 m/sec$^2$. The carried-out dynamic analyses introduce a time discretization step of 0.02 sec.

## NUMERICAL MODELS

A two-dimensional analysis of the structural system is pursued. Thus beams and columns are modeled as 2D beam finite elements, each connecting two nodes. For each element of given length, section area, inertia moment, Young modulus and density, the mass (assumed to be lumped) and the stiffness matrices are obtained [6, 7]. The rotational mass is ignored. The global mass and stiffness matrices are then obtained by assemblage, and the damping matrix follows the Rayleigh scheme [8].

Every node shows three degrees of freedom: horizontal, vertical and rotational. The frame is discretized by 72 nodes (4 levels of 6 nodes each) so that 72 degrees of freedom are computed before the application of the boundary conditions.

The dynamic analysis is driven by the equations:

$$\boldsymbol{M}\ddot{\boldsymbol{d}}(\boldsymbol{t}) + \boldsymbol{G}\dot{\boldsymbol{d}}(\boldsymbol{t}) + \boldsymbol{K}\boldsymbol{d}(\boldsymbol{t}) = \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \tag{1}$$

$$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C_p}\boldsymbol{d}(\boldsymbol{t}) + \boldsymbol{C_v}(\boldsymbol{t})\dot{\boldsymbol{d}}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t}) \tag{2}$$

where $\boldsymbol{M}, \boldsymbol{G}, \boldsymbol{K} \in \mathbb{R}^{n\,x\,n}$ are the mass, damping and stiffness matrices, respectively; $\boldsymbol{B} = -\boldsymbol{M}\boldsymbol{a_g} \in \mathbb{R}^{n\,x\,m}$ is the matrix of the input quantities; $\boldsymbol{C_p}, \boldsymbol{C_v} \in \mathbb{R}^{p\,x\,n}$ are the matrices of the observed variables, to be applied to $\boldsymbol{d}(\boldsymbol{t})$ and $\dot{\boldsymbol{d}}(\boldsymbol{t})$, respectively. In the case under investigation $\boldsymbol{M}$ is diagonal (and therefore invertible).

Let $\tilde{\boldsymbol{z}}(\boldsymbol{t}) = \{\boldsymbol{d}(\boldsymbol{t}), \dot{\boldsymbol{d}}(\boldsymbol{t})\}^{\boldsymbol{T}} \in \mathbb{R}^{2n}$. By introducing the following matrices,

$$\widetilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{G} \end{bmatrix}, \widetilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I} & -\boldsymbol{B} \end{bmatrix}, \ \widetilde{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{G} \end{bmatrix} \quad (3)$$

one obtains:

$$\dot{\boldsymbol{z}}(\boldsymbol{t}) = \widetilde{\boldsymbol{A}}\boldsymbol{z}(\boldsymbol{t}) + \widetilde{\boldsymbol{B}}\boldsymbol{u}(\boldsymbol{t}) \quad (4)$$

$$\boldsymbol{y}(\boldsymbol{t}) = \widetilde{\boldsymbol{C}}\boldsymbol{z}(\boldsymbol{t}) + \widetilde{\boldsymbol{D}}\boldsymbol{u}(\boldsymbol{t}) \quad (5)$$

The boundary conditions deleted 18 (6 by 3) degrees of freedom, so that the square matrices in (1-2) are of size 54, while those in (4-5) are of size 108.

## STANDARD REDUCTION STEPS

The model (say A) described by either Eqs.(1-2) or (4-5) and square matrices of size 54 and 108, respectively, is regarded as the full model for the problem under investigation. A standard Ritz reduction technique is then used to consider the horizontal translation of all the nodes of a same level as slave of the one of the master node defined for that specific level. By applying this method to 3 levels of the frame, so that six translational degrees of freedoms condense into one, a reduction of 15 degrees of freedom is achieved. For this reduced model (say B), the sizes of the square matrices in Eqs. (1-2) and (4-5) are 39 and 78, respectively.

In the absence of rotational inertias, a static condensation of the rotational degrees of freedom is easily achieved at a stiffness level. Since the rotational degrees of freedom are 18, a new reduced order model (say C) is achieved with 21 degrees of freedom or equivalently 42 states.

## REDUCTION BY BALANCED TRANSFORMATION

From the model of Eqs. (4-5), the Gramian matrices of controllability and observability, $\boldsymbol{W_c}$ and $\boldsymbol{W_o}$, satisfy the following pair of Lyapunov equations:

$$\widetilde{\boldsymbol{A}}\boldsymbol{W_c} + \boldsymbol{W_c}\widetilde{\boldsymbol{A}}^{\boldsymbol{T}} + \widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{B}}^{\boldsymbol{T}} = \boldsymbol{0} \quad (6)$$

$$\widetilde{\boldsymbol{A}}^{\boldsymbol{T}}\boldsymbol{W_o} + \widetilde{\boldsymbol{A}}\boldsymbol{W_o} + \widetilde{\boldsymbol{C}}^{\boldsymbol{T}}\widetilde{\boldsymbol{C}} = \boldsymbol{0} \quad (7)$$

respectively. The steps to perform a reduction by balanced transformation are described as follows.

1) Find the Gramian matrices $\boldsymbol{W_c}$ and $\boldsymbol{W_o}$ as solutions of the Lyapunov equations ;

2) Perform the Cholesky factorizations of the Gramian matrices:

$$\boldsymbol{W_c} = \boldsymbol{L_c L_c^T} \;\; ; \;\; \boldsymbol{W_o} = \boldsymbol{L_o L_o^T} \tag{8}$$

3) Consider the Singular Values Decomposition (SVD) of the Cholesky factors:

$$\boldsymbol{L_o^T L_c} = \boldsymbol{U \Lambda V^T} \tag{9}$$

where, denoted the r.h.s. by $\boldsymbol{Q}$, $\boldsymbol{U}$ is the matrix of the eigenvectors of $\boldsymbol{QQ}^T$, $\boldsymbol{V}$ is the matrix of the eigenvectors of $\boldsymbol{Q}^T\boldsymbol{Q}$ and $\boldsymbol{\Lambda}$ is the diagonal matrix of the singular values.

4) Define the balanced transformation:

$$\boldsymbol{T} = \boldsymbol{L_c V \Lambda^{-1/2}} \, , \boldsymbol{T^{-1}} = \boldsymbol{\Lambda^{-1/2} U^T L_o^T} \tag{10}$$

5) Build the state space matrix representation by introducing the matrices:

$$\boldsymbol{A} = \boldsymbol{T^{-1} \widetilde{A} T} = \boldsymbol{\Lambda^{-1/2} U^T L_o^T \widetilde{A} L_c V \Lambda^{-1/2}} \tag{11}$$

$$\boldsymbol{B} = \boldsymbol{T^{-1} \widetilde{B}} = \boldsymbol{\Lambda^{-1/2} U^T L_o^T \widetilde{B}} \tag{12}$$

$$\boldsymbol{C} = \boldsymbol{\widetilde{C} T} = \boldsymbol{\widetilde{C} L_c V \Lambda^{-1/2}} \tag{13}$$

The Hankel singular values are then introduced to operate a model reduction. The starting model is characterized by 42 states, then 3 reduced order models made of 6, 4 and 2 states are achieved.

## OBSERVED VARIABLES

Moving from the state variables to the observed variables, the following relation holds

$$[\boldsymbol{y}] = [\boldsymbol{C}][\boldsymbol{z}] \tag{14}$$

where attention can be focused on accelerations and velocities at given locations

$$\begin{Bmatrix} \boldsymbol{a_t} \\ \boldsymbol{v_t} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{C_a} \\ \boldsymbol{C_v} \end{bmatrix} \{\boldsymbol{z}\} \tag{15}$$

To maintain a symmetric formulation, one divides $\{\boldsymbol{z}\}$ into two vectors $\{\boldsymbol{z_1}\}$ and $\{\boldsymbol{z_2}\}$ such that

$$\begin{bmatrix} a_t \\ v_t \end{bmatrix} = \begin{bmatrix} [C_{a11}] & [C_{a12}] \\ [C_{v11}] & [C_{v12}] \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (16)$$

After the kinematic reductions, the number of state variables, initially in the number of 108, reduced to 78 first and 42 at the end of the last stage The reduction of corresponding $[\boldsymbol{C}]$ matrix is trivial

The further reductions by balanced transformation see the $[\boldsymbol{C}]$ matrix reduced according to Eq.(13). For each of the three studied reduced models (model of 6, 4 and 2 state variables respectively), the following algebra applies:

$$[\boldsymbol{z_1}] = [\boldsymbol{C_{a11}}]^{-1}\{[\boldsymbol{a_t}] - [\boldsymbol{C_{a12}}][\boldsymbol{z_2}]\} \quad (17)$$

$$[\boldsymbol{v_t}] = [\boldsymbol{C_{v11}}][\boldsymbol{z_1}] + [\boldsymbol{C_{v12}}][\boldsymbol{z_2}] \quad (18)$$

$$[(\boldsymbol{a_{t+1}} - \boldsymbol{a_t})\boldsymbol{\Delta t}] = [\boldsymbol{C_{v11}}][\boldsymbol{C_{a11}}]^{-1}[\boldsymbol{a_t}] - [\boldsymbol{C_{v11}}][\boldsymbol{C_{a11}}]^{-1}[\boldsymbol{C_{a12}}][\boldsymbol{z_1}] + [\boldsymbol{C_{v12}}][\boldsymbol{z_2}]$$

From the last equation, after re-arranging , one obtains:

$$[\boldsymbol{z_2}]\{[\boldsymbol{C_{v12}}] - [\boldsymbol{C_{v11}}][\boldsymbol{C_{a11}}]^{-1}[\boldsymbol{C_{a12}}]\} = [(\boldsymbol{a_{t+1}} - \boldsymbol{a_t})\boldsymbol{\Delta t}] - [\boldsymbol{C_{v11}}][\boldsymbol{C_{a11}}]^{-1}[\boldsymbol{a_t}]$$

and eventually

$$\begin{cases} [\boldsymbol{z_2}] = \{-[\boldsymbol{C_{v11}}][\boldsymbol{C_{a11}}]^{-1}[\boldsymbol{C_{a12}}] + [\boldsymbol{C_{v12}}]\}^{-1}\{[(\boldsymbol{a_{t+1}} - \boldsymbol{a_t})\boldsymbol{\Delta t}] - [\boldsymbol{C_{v11}}][\boldsymbol{C_{a11}}]^{-1}[\boldsymbol{a_t}]\} \\ [\boldsymbol{z_1}] = [\boldsymbol{C_{a11}}]^{-1}\{[\boldsymbol{a_t}] - [\boldsymbol{C_{a12}}][\boldsymbol{z_2}]\} \end{cases} \quad (19)$$

## SENSOR LOCALIZATION

It is assumed that the 78 state variable model produces, at any floor, the accelerations which can be measured by suitable accelerometers.

Using this information Eq.s (19) provides an estimate of the state variables for the reduced models characterized by 6,4 and 2 state variables respectively. The estimate differs from the actual model (see Figure 2, as an example).

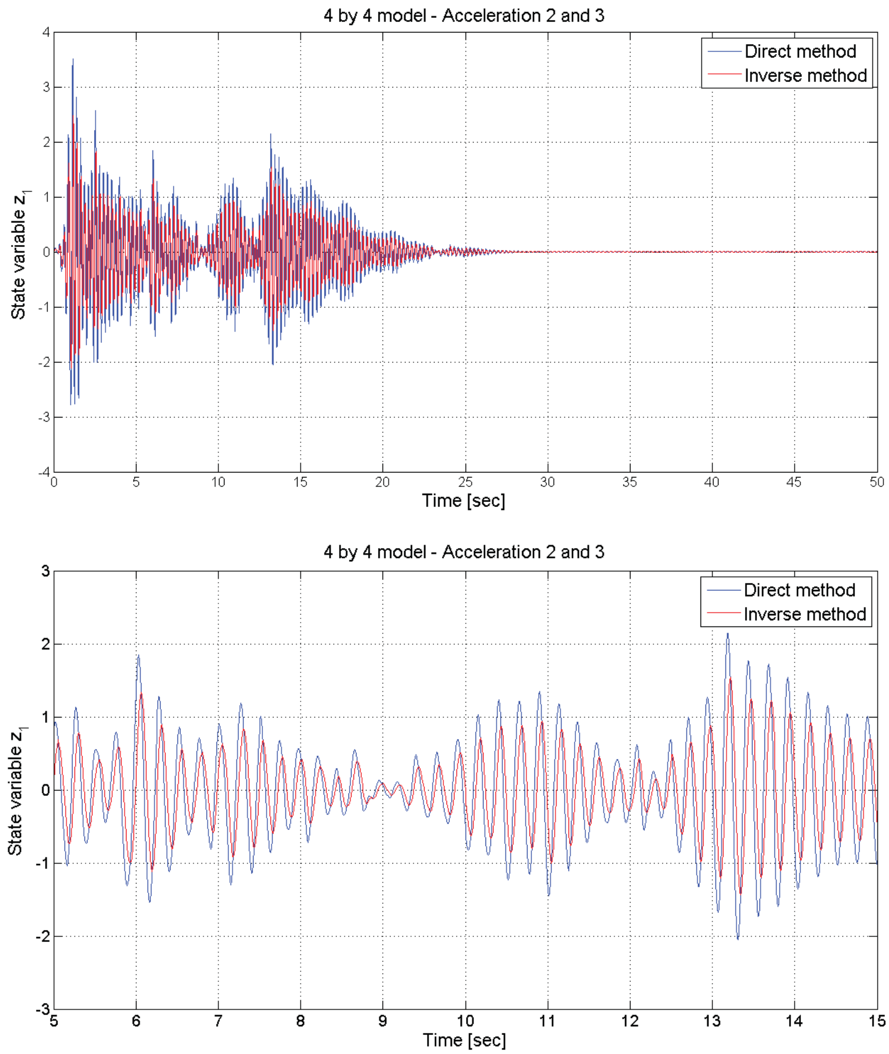


Figure 2 – 4 state variables models associated with combination of acceleration 2 and 3: full time histories (top) and zoom (bottom)

The deviation of these estimates form the state variables as computed by the models themselves can be evaluated by the corresponding sum-of-squares or by its square-root:

$$\boldsymbol{\delta}_{\boldsymbol{i}} = \sqrt{\frac{\sum_{i=1}^{T} (\boldsymbol{z}_{\boldsymbol{i}} * - \hat{\boldsymbol{z}}_{\boldsymbol{i}} *)^2}{N}}$$

where the "*" denotes that each quantity is made dimensionless by dividing its value by the maximum value in the actual time history.

Table 1 – Values of the quantities $\delta_i$ for different scenarios of sensor localization.

| Standard deviation | 6 by 6 model | 4 by 4 model | | | 2 by 2 model | | |
|---|---|---|---|---|---|---|---|
| | 1,2,3 | 1,2 | 1,3 | 2,3 | 1 | 2 | 3 |
| $\delta_{z1,1}$ | 0.0018 | 0.0276 | | 0.0062 | 0.0673 | | |
| $\delta_{z2,1}$ | 0.0018 | 0.0276 | | 0.0062 | 0.0673 | | |
| $\delta_{z1,2}$ | 0.0018 | 0.0276 | **0.0015** | | | 0.0203 | |
| $\delta_{z2,2}$ | 0.0018 | 0.0276 | **0.0015** | | | 0.0203 | |
| $\delta_{z1,3}$ | 0.0018 | | **0.0015** | 0.0062 | | | **0.0173** |
| $\delta_{z2,3}$ | 0.0018 | | **0.0015** | 0.0062 | | | **0.0173** |

Table 1 summarizes the results achieved for the following scenarios

- 6 state-variables model associated with the presence of 3 sensors, one at each floor;
- 4 state-variables model associated with all the combinations of sensors at 2 different floors;
- 2 state-variables model associated with the different location of a single sensor.

If the best location of the sensor is assumed to be the location which minimizes the sum-of-squares, one achieves the sensor locations emphasized in bold in Table 1.

## CONCLUSIONS

Using Model Order Reduction theory [1], this paper shows, with reference to a single case study, how one can benefit of the theory to solve the problem of sensor placement in a system monitoring architecture.

In particular, the use of a reduced number of sensors is associated with a reduced model and the optimal location of the sensors is pursued as the one which minimizes the deviations, from the model state variables time histories, of their estimates.

## ACKNOWLEDGEMENTS

The authors acknowledge the financial support from Catania and Pavia Athenaeum Research grants (PRA and FAR 2011).

## REFERENCES

[1] Schilders W.H.A., van der Vorst H.A., Rommes J.. Model Order Reduction : Theory, Research Aspects and Applications, Springer, 2008

[2] Calberg K., Bou Mosleh C., Farhat C. Efficient non-linear model reduction via a least squares Petrov-Galerkin projection and compressive tensor approximations, *International Journal for numerical methods in engineering*, 86:155-181, 2011

[3] Casciati F., Casciati S., Faravelli L., Franchinotti M. Model Order Reduction vs. Structural Control, ACMA, Fez (Morocco), 2012

[4] Casciati F., Casciati S., Faravelli L., Franchinotti M., Model Order Reduction vs. Structural Control, Proc. 5th ECSC, Genoa, July 2012

[5] Ohtori Y., Christenson R.E., Spencer B.F. Benchmark Control Problems for Seismically Excited Nonlinear Buildings, *J.Eng.Mech.*, 130,366, 2004

[6] Sack R.L., Matrix Structural Analysis. PWS-Kent Pub. Co., Boston (USA), 1989.

[7] Cook R.D., Malkus D.S., Plesha M.E. Concepts and Applications of Finite Element Analysis, John Wiley & Sons, New York (USA), 1989.

[8] Chowdhury, I. Computation of Rayleigh Damping Coefficients for Large Systems, Indian Institute of Technology, The Electronic Journal of Geotechnical Engineering, 2003.