Determination of Stay Cable Force Based on Multiple Vibration Measurements to Consider the Effects of Unsymmetrical Boundary Constraints

W.-H. WU, C.-C. CHEN, M.-R. LEU and G. LAI

ABSTRACT

A novel concept of incorporating the mode shape ratios of cable was recently introduced to develop an accurate method for the determination of cable forces. In this method, a key issue in the optimization process of effective vibration length was to describe the sensor locations by selecting the pre-known middle point of cable as the reference origin point. With this choice, it is equivalent to assume the symmetry of mode shape functions with respect to the middle point of cable. In other words, a crucial restriction of practically symmetric boundary constraints at both ends is imposed. To deal with such difficulties, this method is further generalized in the current study by introducing additional shifting parameters of origin point to effectively consider the unsymmetrical boundary constraints. Several numerical problems of the more complicated nonlinear optimization process associated with this new formulation are first discussed, followed by verifications with extensive numerical examples.

INTRODUCTION

An accurate estimation of stay cable forces typically play an important role in the health monitoring of cable-supported bridges. Due to its simplicity, the ambient vibration method is commonly adopted by first identifying the cable frequencies from the vibration measurements. With given vibration length and flexural rigidity, an analytical or empirical formula is then used with these cable frequencies to determine the cable force. To improve the accuracy of the ambient vibration method, an appropriate selection of parameter values to truthfully reflect the actual vibration behavior is particularly important. In practical cases, rubber constraints and special

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Wen-Hwa Wu, Chien-Chou Chen, Min-Ray Leu, and Gwolong Lai, National Yunlin University of Science and Technology, No. 123, Sec. 3, University Road, Touliu, Yunlin 640, Taiwan (R.O.C.).

anchorage designs are usually installed near both ends of stay cables. These devices, however, significantly increase the uncertainty of boundary conditions and complicate the choice of effective vibration length, which is generally the most sensitive parameter to determine the cable force. Furthermore, each stay cable made by separately arranged steel strands is normally encased in an HDPE tube filled with flexible grouting material to resist corrosion. This situation also induces great difficulties in correctly obtaining the cross-sectional area moment of inertia and the subsequent flexural rigidity of cable.

Aimed to tackle the above problems, a novel concept of incorporating the mode shape ratios of cable was recently introduced by the authors to develop a convenient and accurate method for the determination of cable forces [1-2]. Multiple synchronized vibration signals of a stay cable were first processed to obtain the mode shape ratios at various sensor locations for each observable mode. These ratios were then compared with the sinusoidal mode shapes based on the simply-supported beam model with axial tension to independently obtain an optimal effective vibration length such that the total squares error for all the considered modes is minimized. With this length obtained, the cable force and flexural rigidity can subsequently be solved by simple linear regression techniques using the identified cable frequencies and the analytical formula. In this method, a key issue in the optimization process of effective vibration length was to describe the sensor locations by selecting the pre-known middle point of cable as the reference origin point for the sinusoidal shape functions. With this choice, it is equivalent to assume the symmetry of mode shape functions with respect to the middle point of cable. In other words, a crucial restriction of practically symmetric boundary constraints at both ends is imposed with this mathematical formulation.

Even though the cable anchorage systems in most of the practical designs may not be far away from this simplification, there certainly exist a number of cases with apparent unsymmetrical boundary constraints, especially when supplementary dampers are installed at the deck ends of stay cables. To deal with such difficulties, this method is further generalized in the current study by introducing additional shifting parameters of origin point in the sinusoidal shape functions to effectively consider the unsymmetrical boundary constraints. Several numerical problems of the more complicated and sensitive nonlinear optimization process associated with this new formulation are first discussed in this paper, followed by verifications with extensive numerical examples.

STAY CABLE AND ITS ANALYSIS WITH A SIMPLIFIED MODEL

A stay cable system can be typically divided into three parts: (1) a free length section in the middle; (2) two anchorage zones at both end; and (3) two transition zones between the previous two parts. The combination of the anchorage zone and the transition zone is usually called the cable anchorage device, whose detailed design varies with the suppliers. But in general, flexible rubber constraints are installed at the front end of anchorage device to reduce the bending stress at anchorage ends induced by lateral cable vibrations, centralize the cable, alleviate the fatigue problem, and additionally provide certain amount of damping. Because of the complicated anchorage device, it is difficult to accurately define the boundary conditions and effectively model the sections close to both ends in performing the cable analysis. Nonetheless, it is noteworthy that the effect of anchorage device on the cable vibration should be limited

in a finite range near the anchorage ends. Thus, the primary free length section in the middle of cable ought to be eligibly modeled by a simply supported beam with an axial tension. The only key problem is how to select an effective length for this model.

Considering a simply supported beam subjected to an axial tension T, an analytical formula for the modal frequencies of this model can be solved as:

$$\left(\frac{f_k}{k}\right)^2 = \frac{T + \dfrac{k^2\pi^2\hat{E}I}{L^2}}{4\overline{m}L^2} \quad \text{or} \quad T = 4\overline{m}L^2\left(\frac{f_k}{k}\right)^2 - \frac{k^2\pi^2\hat{E}I}{L^2}, \quad k = 1,\ 2,\ 3,\ \cdots \tag{1}$$

where L signifies the beam length, $\overline{m}$ symbolizes the mass per unit length, $\hat{E}$ denotes the Young's modulus, I represents the cross-sectional area moment of inertia, and f_k is the natural frequency of the k-th mode in Hz. Moreover, the mode shape corresponding to each modal frequency f_k is found to be in the form of sinusoidal functions:

$$\sin\frac{k\pi x}{L}, \quad k = 1,\ 2,\ 3,\ \cdots \tag{2}$$

METHODOLOGY

It is especially noteworthy in Eq. (2) that this set of modal shapes sorely depends on the vibration length L. This provides a significant contrast to Eq. (1) where several parameters are involved. Even if $\overline{m}$ can be considered a known value because of its reliable estimation in practical applications, it is obvious from Eq. (1) that each modal frequency is still a function of T, L, and the flexural rigidity $\hat{E}I$. In other words, all the above three unknown parameters are coupled if only the modal frequencies are available. An enlightening clue disclosed from Eq. (2) is that the effective vibration length of cable can be independently determined as long as the information of modal shape functions is accessible. With this obtained effective vibration length, each modal frequency turns out to be simply a linear function of the cable force and the flexural rigidity. Therefore, the optimal values for the two remaining unknown quantities can then be solved from the identified modal frequencies utilizing the least squares method.

It is unavoidable to conduct multiple synchronized measurements for estimating the mode shape ratios. Assume that $y(x_1,\ t)$, $y(x_2,\ t)$, $\cdots$, $y(x_n,\ t)$ are n signals simultaneously measured from n different locations of the same cable and only the m most significant modes with major contribution are considered. With these measurements, the mode shape vector at the n measured points for the k-th mode can be estimated from the Fourier transforms $Y(x_1,\ \omega)$, $Y(x_2,\ \omega)$, $\cdots$, $Y(x_n,\ \omega)$ of measurements at $\omega = \omega_k$ and expressed as:

$$\hat{\boldsymbol{\phi}}_k \equiv \left\{\begin{array}{c} \hat{\phi}_k(x_1) \\ \vdots \\ \hat{\phi}_k(x_n) \end{array}\right\} \equiv \left\{\begin{array}{c} \hat{\phi}_{1k} \\ \vdots \\ \hat{\phi}_{nk} \end{array}\right\} \approx \left\{\begin{array}{c} Y(x_1,\omega_k)/Y(x_i,\omega_k) \\ \vdots \\ Y(x_n,\omega_k)/Y(x_i,\omega_k) \end{array}\right\}, \quad k = k_1, k_2, \cdots, k_m \tag{3}$$

where $k_1, k_2, \cdots$, and k_m stand for the mode orders of the m major modes, respectively. Since the mode shape ratios are theoretically real, the real parts of the estimated values from Eq. (3) are taken as the mode shape ratios and their trivial imaginary parts cab be

used to indicate the effectiveness of these measurements. It should be also noted that any one of $Y(x_1, \omega_k)$, $Y(x_2, \omega_k)$, $\cdots$, $Y(x_n, \omega_k)$ can be taken as the common denumerator $Y(x_i, \omega_k)$ in Eq. (3).

The sinusoidal shape functions in Eq. (2) are obtained by setting the origin point at one end of the beam model to create a range of $0 \le x \le L$ for the independent variable. However, the vibration length of model and consequently the corresponding boundary points are left open to be determined in the current case of cable force estimation. To effectively describing the measurement locations, the authors recently proposed [1-2] an origin shift to the middle point between the front edges of rubber constraints at both ends, which can be decided without knowing the vibration length in advance. With this coordinate transformation coming from the assumption of symmetric anchorages at both ends, the even mode shapes remain as sine functions, but the odd mode shapes turn into cosine functions, both falling in the range of $-L/2 \le x \le L/2$. In other words, the theoretical mode shape vector $\boldsymbol{\phi}_k$ can be expressed in the interval $-L/2 \le x \le L/2$ as:

$$\boldsymbol{\phi}_k = \begin{Bmatrix} \phi_k(x_1) \\ \vdots \\ \phi_k(x_n) \end{Bmatrix} = a_k \begin{Bmatrix} \cosin \dfrac{k\pi x_1}{L} \\ \vdots \\ \cosin \dfrac{k\pi x_n}{L} \end{Bmatrix}, \quad k = k_1, k_2, \cdots, k_m \tag{4}$$

where a_k denotes the amplitude coefficient of the k-th mode and the function $\cosin(\cdot)$ is defined by

$$\cosin(\cdot) = \begin{cases} \cos(\cdot) & \text{if } k = \text{odd} \\ \sin(\cdot) & \text{if } k = \text{even} \end{cases} \tag{5}$$

An appropriate error function has to be defined as the objective function before the optimization procedures can be performed. In the current case, the optimization problem for determining the effective vibration length of cable is to search for the optimal value of L such that the error by comparing the estimated mode shape ratios of Eq. (3) in all the m major modes with their corresponding values from theoretical mode shape functions can be minimized. Therefore, the objective error function for optimization was previously defined [1-2] as:

$$E = \sum_{k=k_1}^{k_m} \sum_{j=1}^{n} \left(\phi_{jk} - \hat{\phi}_{jk} \right)^2 = \sum_{k=k_1}^{k_m} \sum_{j=1}^{n} \left[a_k \cosin\left(\frac{k\pi x_j}{L} \right) - \hat{\phi}_{jk} \right]^2 \tag{6}$$

It should be noticed that there are $m+1$ unknown coefficients in Eq. (6) including m different amplitude coefficients a_k and L. Furthermore, this is a nonlinear optimization problem because L appears in the denumerator of cosin function.

To generalize the previously proposed symmetric formulation, an origin shifting parameter d_k can be additionally introduced such that each sinusoidal shape function is free to move toward either anchorage end. Furthermore, another adjustment is also made in this study by assigning an independent vibration length L_k for each mode to create more flexibility in matching the mode shape ratios. With these two modifications, the total error function for optimization is redefined as:

$$E = \sum_{k=k_1}^{k_m} \sum_{j=1}^{n} \left\{ a_k \cosin\left[\frac{k\pi(x_j - d_k)}{L_k} \right] - \hat{\phi}_{jk} \right\}^2 = \sum_{k=k_1}^{k_m} e_k \tag{7}$$

where

$$e_k \equiv \sum_{j=1}^{n} \left\{ a_k \cosin\left[\frac{k\pi(x_j - d_k)}{L_k} \right] - \hat{\phi}_{jk} \right\}^2 , \quad k = k_1, k_2, \cdots, k_m \tag{8}$$

and the positive value of d_k indicates that the corresponding shape function shifts towards the right end. It needs to be particularly noted that there are totally $3m$ unknown coefficients in Eq. (7) compared to $m+1$ of them in Eq. (6). As shown in Eq. (8), however, the three unknown parameters for each mode are independent of those of the other modes and can be obtained from the error function e_k for each individual mode. Therefore, the more parameters with this new formulation would not increase the computational load in the optimization process. Instead, separation of effective lengths for various modes would greatly improve the efficiency of this methodology.

With the optimal length obtained and the identified modal frequencies already known, Eq. (1) becomes a linear function of the cable force and the flexural rigidity:

$$T + \left(\frac{k^2 \pi^2}{L^2} \right) \hat{E}I = 4\overline{m}L^2 \left(\frac{f_k}{k} \right)^2 \tag{9}$$

To decide these two remaining unknown quantities, consistent optimization procedures by considering the same m modes used in determining the vibration length are suggested. More specifically, Eq. (1) is rearranged into the form of a classical linear regression problem such that T and $\hat{E}I$ can be conveniently.

NUMERICAL VERIFICATION

The newly developed methodology is applied to analyze the stay cables of Chi-Lu Bridge. This bridge is a two-span (120m+120m) cable-stayed bridge connecting the two towns Chi-Chi and Lu-Ku located in central Taiwan. There are totally 34 pairs of stay cables installed on Chi-Lu Bridge. SAP2000 software is adopted in this section to construct the finite element models for the longest cable R33 and shortest cable R01 with input parameters listed in Table 1 to numerically verify the feasibility and accuracy of the proposed method. The model for each cable is composed of 500 frame elements of equal length together with two linear spring elements located at the front end locations of the actual rubber constraints, as shown in Fig. 1.

With the FE models prescribed above, the corresponding modal frequencies and mode shape vectors can be obtained by running the modal analysis in SAP2000 and are then regarded as the identified modal parameters in practical applications. To imitate

Table 1. Input parameters for FE models of Cables R01 and R33.

Cable No.	Total Length (m)	Length between Springs (m)	Mass per Unit Length $\overline{m}$ (kg/m)	Flexural Rigidity $\hat{E}I$ (MN·m^2)	Cable Force T (MN)
R01	29.30	23.06	61.30	0.868	3.25
R33	126.42	118.26	48.00	0.531	2.14

3.25MN L=29.303 m   3.25MN

L_1=3.5 m L_2=2.7 m

(a) Cable R01

2.14MN L=126.42 m   2.14MN

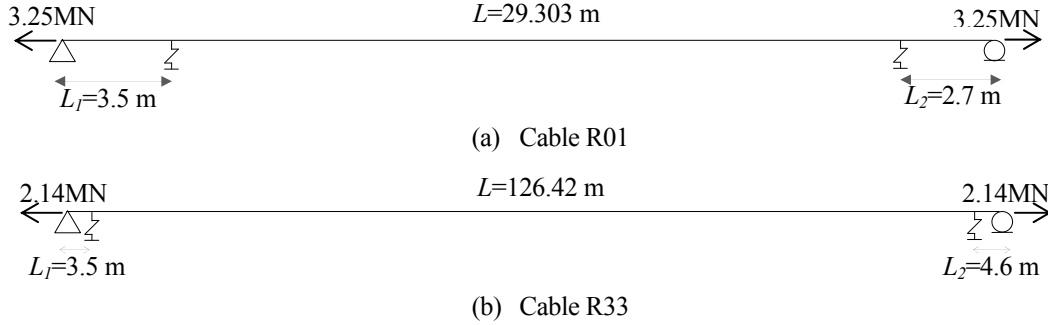L_1=3.5 m L_2=4.6 m

(b) Cable R33

Figure 1. Finite element models of Cables R01 and R33.

the practical situations where only a few measurements close to the deck end can be conveniently taken, the mode shape ratios at 3 nodes corresponding to accessible measurement locations are chosen for determining the effective vibration length. It is also assumed that the 2nd, 3rd, and 4th mode are the major contribution modes in each case. In other words, $n = 3$, $m = 3$, $k_1 = 2$, $k_2 = 3$, and $k_3 = 4$ are adopted for optimization in the numerical examples.

Although Eq. (7) provides a generalized formulation to deal with unsymmetrical boundary conditions and holds the advantage in portioning the optimization into each mode, it is also associated with a major numerical difficulty regarding the convergence in the optimization process. Compared to the symmetrical formulation of Eq. (6) with which a rapid and smooth convergence can be attained by any reasonable initial guess [1-2], the unsymmetrical formulation with the addition of shifting parameter d_k results in a much more complicated optimization problem extremely sensitive to the initial guesses. To illustrate this difficulty, the error function E for the case of Cable R01 with $m = 1$ and $k_1=1$ is plotted in Fig. 2 over a practically possible range of $d =d_1$ and $L =L_1$ under a prescribed value of a_1. It is clear that E is basically a smooth function of d and L with a canyon shape, but its value abruptly drops along a slanted straight line on the d-L plane. Closer examination along this deep trench reveals its remarkable roughness, especially in the neighborhood of the global minimum. In other words, the global minimum is surrounded by numerous local minima. Accordingly, the optimization processes starting with different initial guesses could end up with converging to diverse local minima and the global minimum is particularly difficult to
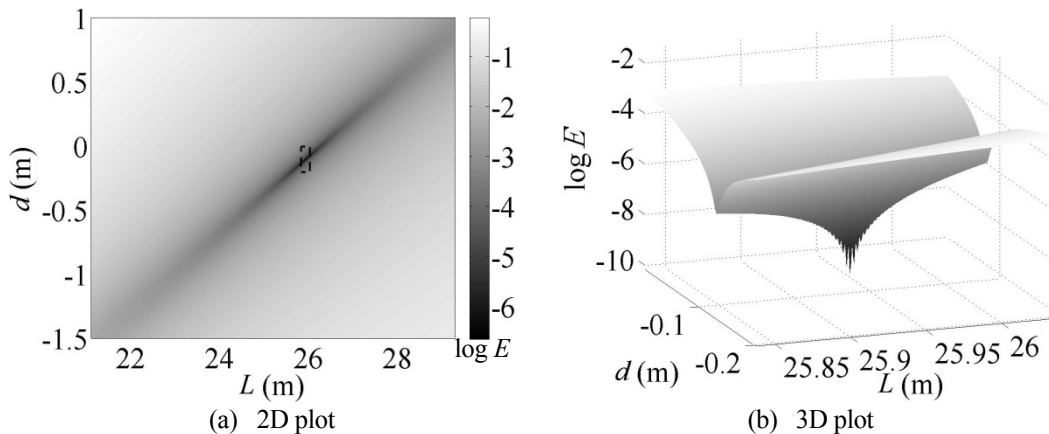

(a) 2D plot   (b) 3D plot

Figure 2. Total square error function under different values of parameters.

reach. But on the other hand, Fig. 2 also suggests that this problem may not be a fatal one in the sense of engineering applications. Since all the local minima including the global minimum resides in a very narrow region on the d-L plane, the convergent values of d and L obtained from optimization with various initial guesses should all be close and essentially make no difference.

According to the above investigation, two key features are particularly important for developing a feasible approach to crack the numerical puzzle associated with the unsymmetrical formulation. First, the convergence complexity is induced by adding the shifting parameter d_k and a simple optimization problem similar to that resulted from the symmetric formulation can be resumed if d_k is removed out of the optimization process. Besides, keeping the initial guesses for d and L in the deep trench range would significantly facilitate the convergence rate. Consequently, an efficient algorithm is established in this study by minimizing Eq. (7) with given values of d_k. In this case, the optimization problem is reduced to that of Eq. (6) and the optimal parameter values need to be decided from different trials of d_k. Because the reasonable values of d_k are typically limited to the range between $-L_1/2$ and $L_2/2$ as shown in Fig. 1, the computational cost for trying different values of d_k in this range can be kept in an acceptable manner. Furthermore, at least two values of d_k (say, 0 and other possible values from engineering judgments) can be tried first to obtain the corresponding sub-optimal values of L_k. With these few sub-optimal pairs of d_k and L_k available, a straight line on the d-L plane is then easily determined by linear regression to locate the deep trench as shown in Fig. 1. For the remaining trials of d_k, the initial guess of L_k decided by the regressed line would certainly accelerate the optimization process.

To verify the convergence of the proposed optimization algorithm, the results for the case of Cable R01 with both spring coefficients equally taken as $K_s = 10^4$, 10^6, and 10^8 N/m are listed in Table 2. In this table, the values of optimal parameters and error function determined by simulated shape ratios of the first mode are compared for three different divisions of d between $-L_1/2$ and $L_2/2$. As expected, a smaller value of total error E closer to the global minimum is obtained with a larger number of divisions in all the cases. More importantly, the optimal values for the effective length and shifting parameter are slightly different with either 20, 40, or 80 divisions of d, but all with a negligible variation as previously discussed. In other words, 20 trials of d between $-L_1/2$ and $L_2/2$ should be sufficient in this case. In fact, the corresponding results for the case of Cable R33 share exactly the same trend. To be more conservative, the results in the following analysis are all obtained under 80 trials of d.

The values of effective vibration length and shifting parameter determined for

Table 2. Optimal parameters and errors of Cable R01 determined by simulated shape ratios of 1st mode.

Spring Coefficient K_s (N/m)	Effective Length with Different Divisions of d (m)			Shifting Parameter with Different Divisions of d (m)			Total Error E with Different Divisions of d (10^{-10})		
	20	40	80	20	40	80	20	40	80
10^4	28.07	28.28	28.17	−0.47	−0.39	−0.43	93.0	23.7	5.50
10^6	26.03	26.03	25.92	−0.06	−0.06	−0.11	56.1	56.1	1.55
10^8	22.74	22.74	22.74	0.06	0.06	0.06	5.31	5.31	5.31

Table 3. Optimal parameters of Cable R01 determined by simulated shape ratios of different modes.

Spring Coefficients K_s (N/m)	Effective Length for Different Modes (m)			Shifting Parameter for Different Modes (m)		
	2nd	3rd	4th	2nd	3rd	4th
$10^6 + 10^6$	26.03	26.22	26.54	−0.14	−0.18	−0.23
10^5 (L)+10^7 (R)	25.95	26.05	26.08	−1.36	−1.37	−1.41

Table 4. Tensions of Cables R01 and R33 determined by 3 simulated mode shape ratios.

Spring Coefficients K_s (N/m)	Formulation	Tension T of Cable R01 (10^6N)			Tension T of Cable R33 (10^6N)		
		Given Value	Estimated Value	Error (%)	Given Value	Estimated Value	Error (%)
$10^6 + 10^6$	Unsymmetrical	3.25	3.25	0.14	2.14	2.15	0.34
	Symmetric		3.66	12.6		2.14	−0.18
10^5 (L)+10^7 (R)	Unsymmetrical		3.25	0.06		2.14	0.09
	Symmetric		5.30	63.2		2.25	5.33

different modes (2nd, 3rd, and 4th) of Cable R01 are arranged in Table 3 where a situation with both spring coefficients equally taken as 10^6 N/m is compared with the other with a left spring of 10^5 N/m and a right spring of 10^7 N/m. It is apparent that the optimal parameters determined for different modes are basically consistent. Particular attention needs to be paid on the optimal values of shifting parameter. A minor negative value of shifting parameter in the case with even springs is caused by the slight asymmetry in geometry ($L_1 = 3.5$m $> L_2 = 2.7$m), while a larger negative value in the other case truthfully reflect the effect of uneven springs.

With the obtained optimal parameters, the given modal frequencies for the 3 chosen modes can be substituted into Eq. (9) to solve the cable force and flexural rigidity. The results for Cables R01 and R33 are listed in Table 4, from which a few enlightening conclusions can be made. The results based on the previously developed symmetric formulation are also presented for contrast. It is first noted that the accuracy of the newly proposed method in estimating the cable force is superb with errors far less than 1% in all the cases. On the other hand, the error for the case of the short cable R01 using the symmetric formulation reaches 13% when small asymmetry in geometry occurs and can even go as high as 63% if the effect of uneven springs is also involved. In the cases for the long cable R33, the benefit of applying the new method is not as significant, obviously due to the relatively trivial influence range of boundary conditions for the long cable. However, the use of symmetric formulation in the situation of uneven springs still induce more than 5% of error, which can be successfully reduced to less than 0.1% with the unsymmetrical formulation.

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