A Loading Basis for Plate Structure Under Tension Loads and Application to Full-Field Reconstruction

D. MARTINI, CH. HOCHARD and J-P. CHARLES

ABSTRACT

This method defines a loading basis for plate structures which is identified from strain measurements, in order to reconstruct the mechanical fields. This loading basis is given by a decomposition of the global structure into simple sub-structures associated with the loaded boundaries only. Some elementary basis are defined for each substructure depending on their local edge effects. A global basis is then obtained by the equilibrium of the complete structure. The main advantage of this approach is to classify the basis vectors depending on their influence on the overall response of the structure.

INTRODUCTION

Improving structural performances requires monitoring the mechanical fields present inside the structures and the boundary conditions. Assuming here that the loading zones are not overstressed, overall information about the whole structure are therefore preferred to detailed information about specific small zones. This approach focuses mainly on the overall responses of the structures associated with the boundary conditions. Full-field identification methods have been widely studied for the last ten years in order to improve the control and the performances of stand-alone structures. Various methods have been implemented in the context of both dynamic and static field reconstructions. The dynamic approaches have consisted in identifying modal shapes from strain measurements [3, 6]. The static approaches have consisted in identifying

Martini, Hochard, Charles, Aix-Marseille University (AMU), Laboratory of Mecanics and Acoustics (LMA), Marseille, France

finite element fields from strain and displacement measurements [7, 8]. The main limitation of these methods is the mismatch between the degrees of freedom used to approximate mechanical fields and the few number of measurements available in practical applications. Hundred of thousands of parameters are required to accurately approximate mechanical fields, whereas only a few sensors are available, which results in an ill-posed inverse problem, [4]. Since the boundary conditions are used to regularize parameter identification problems on the basis of full-field measurements [1], it was proposed to first identify the boundary conditions and then to reconstruct the mechanical fields. In addition, only internal fields are studied, and the corresponding boundary conditions are therefore approximated with just a few parameters based on Saint-Venant's principle. These parameters can therefore be identified with a limited amount of measurements, which regularizes the inverse problem associated with the full-field reconstruction problem.

In this paper, the formulation associated with structural monitoring is first tackled and the resolution with a finite element method is proposed for plate structures. The main issue is then to find the proper loading basis, which accurately approximates the real loading conditions. This basis is obtained by the projection of Trefftz-like analytical solutions of the plate problem onto the boundaries of the structures. Lastly, a decomposition of plate structures is used to define a global loading basis. With this method, the loading functions are organized in increasing local edge effects which simplifies the choice of these functions for monitoring the overall responses of the structures.

INVERSE PROLEM FORMULATION

The aim here is to determine mechanical fields and boundary conditions based on strain measurements. Let us take a structure Ω subjected to unknown loading conditions, F_b onto the boundary $\partial\Omega$. At this point, we take the boundary conditions to be loads only, the displacement boundary conditions, u_b relates to the rigid body motion (RBM) only and F_b is necessarily balanced. This assumption, which is required because no displacement measurements are available, makes it possible to avoid the uncertainties about the joints between the structure and the environment. Strain measurements, ε_m, are performed on $\partial\Omega_m \subset \partial\Omega$ and the effects of the body force and the inertia are neglected. Summarizing the inverse problem in equation form gives:

To find $(u, \varepsilon, \sigma, F_b)$ such that:

$$\text{Mechanical equations:} \begin{cases} div\,[\sigma] = 0 \text{ in } \Omega \\ \sigma \cdot n = F_b \text{ in } \partial\Omega_F \\ u = u_b \text{ in } \partial\Omega_u \\ \sigma = \mathbb{C}\varepsilon \text{ in } \Omega \end{cases} \tag{1}$$

$$\text{Observation equation:} \varepsilon = \arg\min \phi\left(\varepsilon - \varepsilon_m\right) \text{ in } \partial\Omega_m \subset \partial\Omega$$

This inverse problem is ill-posed, [4], because many parameters, $DoFs \approx 10^6$, have to be identified based on a few measurements ≈ 20. Based on the classification of inverse problem in [5], we decided to define a regularized inverse problem with only a few parameters. Improving the structural performances involves monitoring the internal fields, which mostly depend on the overall effects of the loading conditions, as shown by Saint-Venant's principle. The internal fields can therefore be properly reconstructed using some approximate loading conditions such that the overall effects of these approximate and real loading conditions are similar. It is now proposed to identify the loading conditions, $\bar{F}_b$ and then to reconstruct the mechanical fields, $(\bar{u}, \bar{\varepsilon}, \bar{\sigma})$ in Ω. The inverse problem is now reduced to:

$$\begin{aligned}
&\text{To find } (\bar{u}, \bar{\varepsilon}, \bar{\sigma}, \bar{F}_b, \bar{u}_b) \text{ such that:}\\
&\text{Mechanical equations: } \begin{cases} div[\bar{\sigma}] = 0 \text{ in } \Omega \\ \bar{\sigma} \cdot n = \bar{F}_b \text{ in } \partial\Omega_F \\ \bar{u} = \bar{u}_b \text{ in } \partial\Omega_u \\ \bar{\sigma} = \mathbb{C}\bar{\varepsilon} \text{ in } \Omega \end{cases}\\
&\text{Observation equation: } \bar{F}_b = \arg\min\left[\|\bar{\varepsilon}(\bar{F}_b) - \varepsilon_m\|^2\right] \text{ in } \partial\Omega_m \subset \partial\Omega
\end{aligned} \tag{2}$$

This problem is then solved with the finite element method. Displacement field u is approximated by $\bar{u}(X) = \sum \Phi_i(X)\bar{U}_i$. The strain field is determined from the strain-displacement relation and the stress field is calculated using the constitutive laws. The loading conditions, F_b, are approximated by a $q-$dimension basis, giving $\bar{F}_b = \sum_{i=0}^{q} f_i F_b^i$ where f_i are the unknown loading parameters and F_b^i are some loading functions defined onto the boundary $\partial\Omega$. Using the gradient operator $\mathbb{B}$ and the projector into $\partial\Omega_m$, Π_m, the discretization of the mechanical equations gives the following expression:

$$\begin{aligned}
&\text{To find } (\bar{U}, \bar{\varepsilon}, \bar{\sigma}, \bar{F}) \text{ such that:}\\
&\text{Observation equation: } \bar{F} = \arg\min_{F\in\mathbb{R}^q} \|\mathbb{B}_{\pi m} U(F) - \varepsilon_m\|^2\\
&\qquad\qquad\qquad\qquad \Leftrightarrow \bar{F} = \arg\min_{F\in\mathbb{R}^q} \|\mathbb{B}_{\pi m}\mathbb{K}^{-1}\mathbb{A}F - \varepsilon_m\|^2\\
&\text{Mechanical equations: } \begin{cases} \bar{\varepsilon} = \mathbb{B}\bar{U} \\ \bar{\sigma} = \mathbb{C}\bar{\varepsilon} \\ \mathbb{K}\bar{U} = \mathbb{A}\bar{F} \Leftrightarrow \bar{U} = \mathbb{K}^{-1}\mathbb{A}\bar{F} \end{cases}
\end{aligned} \tag{3}$$

where $\mathbb{K}$ is the stiffness matrix of the structure, $\bar{U} = ^T[u_1, u_2, \cdots, u_n]$ is the nodal displacement vector and $\bar{F} = ^T\left[f_1, f_2, \cdots, f_{q-1}, f_q\right]$ is the loading parameter vector. The $\mathbb{A}$ matrix is the projection operator associated with each elementary loading function. Lastly, the structural behavior is assumed to be linear, and the observation equation can therefore be directly solved in the least-squares sense with :

$$\bar{F} = \left(^T\mathbb{G} \cdot \mathbb{G}\right)^{-1} \cdot {}^T\mathbb{G} \cdot \varepsilon_m \tag{4}$$

The main advantage of this method is that it avoids the instability occurring when increasing the FE degrees of freedom (DoFs). The identification procedure depends only on the number of loading parameters. The number of DoFs involved in the FE model used to approximate the mechanical fields can therefore be as large as required to ensure the accuracy of the FE model. In addition, as the displacement boundary conditions are fixed (RBM), these mechanical fields can be computed directly, using a linear combination of precomputed solutions associated with each of the vectors in the loading basis.

PLATE STRUCTURES under TENSION LOADS

Construction of the loading basis for plates

A basis focusing on the overall response of the structure was constructed to approximate the loading conditions. This construction was based on the Trefftz-like solutions developed in [9, 2] by approximating the displacement field with polynomial functions satisfying the equilibrium equation. The basis of the loading conditions was obtained from the projections of these Trefftz-like solutions onto the boundaries of the structure. The main advantage of this method is that the approximate displacement fields are a complete set of solutions for a given polynomial order. The projections of these solutions are therefore a complete set of loading conditions the polynomial orders of which are associated with increasing edge effects. Low polynomial orders were therefore used to limit these edge effects in order to focus on the overall response of the structure alone. The corresponding $q-$order set of loading conditions is defined as follows:

$$\begin{aligned} &\mathcal{F}^q = \{F^r = \sigma(\bar{u}^r)\cdot n_{\partial\Omega} \; ; \; \bar{u}^r \in \mathcal{U}^q\} \\ &\text{with } \; \mathcal{U}^q = \left\{ \bar{u}^r = \sum_{j=0}^{r} \alpha_{jr} x^j y^{r-j} \; ; \; 0 < r \leq q \; \& \; div\left[\mathbb{C}\varepsilon(\bar{u}^r)\right] = 0 \; in \; \Omega \right\} \\ &\qquad \text{and } \Omega \text{ is a star-shaped domain} \end{aligned} \tag{5}$$

The vectors of this loading basis are then orthonormalized using the Gram-Schmidt algorithm in the sense of strain energy. The orthonormalized basis $\mathcal{F}_{\perp}^q$ is obtained by:

$$\begin{aligned} \mathcal{F}_{\perp}^q = \Big\{ F_{\perp}^r \in \mathcal{F}^q \; ; \; \forall\, k < r \; \int_{\Omega} Tr\left[\sigma(F_{\perp}^r)\,\varepsilon\left(F_{\perp}^k\right)\right] d\Omega = 0 \; \& \\ \int_{\Omega} Tr\left[\sigma(F_{\perp}^r)\,\varepsilon(F_{\perp}^r)\right] d\Omega = 1 \Big\} \end{aligned} \tag{6}$$

Extension to the plate structures

At this point the main limitation of the proposed method is the restriction of the loading basis to star-shape domain only. A first extension of this basis is obtained for structures with inside stress concentration such as holes or notches. This extension is presented in [10]. We propose now an extension of the loading basis for plate structures under tension loads. This extension consists in substructuring the complete structure into elementary structures which are associated with the boundaries having unknown loads. Trefftz-like loading bases are then defined for each of these substructures and the global equilibrium of the complete structure is used to define a global loading basis. Figure 1 illustrates the definition of these substructures in the case of three boundaries with unknown loads, $\partial\Omega_1$, $\partial\Omega_2$ and $\partial\Omega_3$ of the structure Ω. Noting F_i the unknown loading condition of the boundary $\partial\Omega_i$ and $\left(F_{bi}^k\right)_{1\le k\le p_i}$ the loading basis of the substructure Ω_i, the global equilibrium of the complete structure is expressed by :

$$\left\{\begin{array}{l}\displaystyle\sum_{i=1}^{r}\sum_{k=1}^{p_i}\alpha_k^i F_{bi}^k = 0\\ \displaystyle\sum_{i=1}^{r}\sum_{k=1}^{p_i}\alpha_k^i \; OP\wedge F_{bi}^k = 0\end{array}\right. \Leftrightarrow \mathbb{H}\cdot F = 0 \tag{7}$$

The loading basis $\mathcal{F}_\perp^T$ of the complete structure is then defined by the kernel of $\mathbb{H}$ and orthonormalized with respect to the strain energy :

$$\mathcal{F}_\perp^T = \left\{F_\perp^r \in Ker\mathbb{H}\;;\;\forall\, k<r\;\int_\Omega Tr\left[\sigma(F_\perp^r)\,\varepsilon\left(F_\perp^k\right)\right]d\Omega = 0\;\&\; \int_\Omega Tr\left[\sigma(F_\perp^r)\,\varepsilon(F_\perp^r)\right]d\Omega = 1\right\} \tag{8}$$

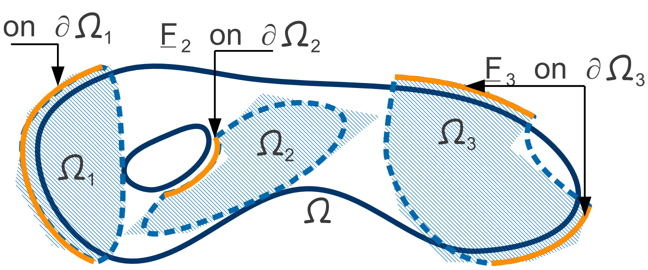

Figure 1: Illustration of the complete plate structure Ω and of the substructures Ω_1, Ω_2 and Ω_3 respectively associated with the boundaries $\partial\Omega_1$ $\partial\Omega_2$ and $\partial\Omega_3$ having unknown loads.

Application to a $L-$structure

Let's consider the $L-$structure Ω. The top edge is fully clamped and two edges on the left are linearly loaded as illustrated in figure 2. These two displacements and loading conditions are supposed to be unknown and associated with the unknown loads F_1 and F_2, and the two substructures Ω_1 and Ω_2 are defined by the corresponding boundaries as shown in figure 2. The loading basis of the complete structure is obtained by the two Trefftz-like bases of each substructure. The sensitivity of the reconstructed fields to error measurements is then studied depending on the sensor locations. The dimension of the global loading basis is 14 and 5 sensors are only needed to identify the loading parameters. Figure 3 shows the distribution of the error of the reconstructed solutions without noise. We observe that the error is minimum in the structure and also at the unloaded boundaries. These results show the advantage of the proposed loading basis that localizes the uncertainties of the model at the

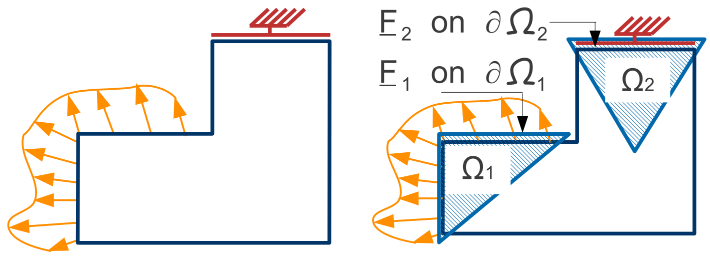

Figure 2: Illustration of the $L-$ structure Ω with its loading conditions and illustration of the elementary substructures associated with the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ having unknown loads F_1 and F_2.

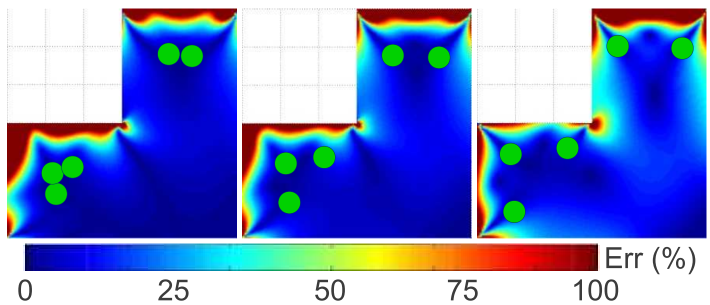

Figure 3: Distributions of the error of the reconstructed solutions with different observation distances and using unnoisy measurements. The green circles show the sensor locations in each substructure Ω_1 and Ω_2.

boundaries with unknown loads only. At the opposite, the " inside" mechanical fields are accurately reconstructed. This result also gives an optimal observation distance depending on the error threshold that is similar to the one-plate case. Lastly, figure 4 shows the distribution of the standard deviation ratio τ_{STD} with different observation distances. The proposed loading basis is therefore a proper approximating basis for the loading conditions because the errors of the reconstructed fields are equivalent to the measurement errors in the structure and the maximum errors are localized at the loaded boundaries.

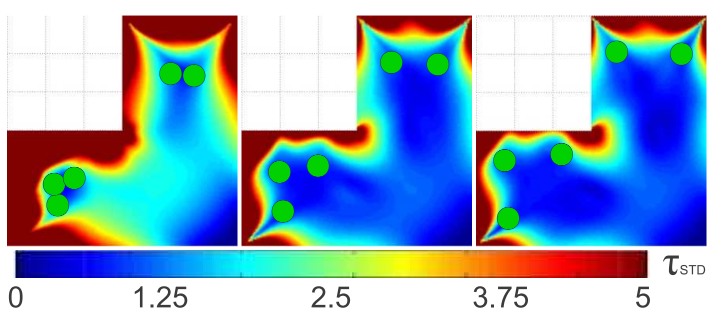

Figure 4: Distributions of the standard deviation ratio τ_{STD} of the reconstructed solutions with different observation distances and using noisy measurements. The green circles show the sensor locations in each substructure Ω_1 and Ω_2.

CONCLUSION and PROSPECTS

In this paper, we showed that the full-field reconstruction could be reduced to a load identification problem, which regularizes this initial issue. Those unknown loads are described with just a few parameters in comparison with the unknown mechanical fields because overall information about the structure are required for structural monitoring purposes such as command and performances. We then proposed a method for defining a loading basis for plates and plate structures in order to reconstruct the mechanical fields from the loading parameters identification. This loading basis is associated with the overall response of the structure based on analytical Trefftz-like solution. Overall information is accurately recovered in the structure because the effects of the measurement errors are localized at the boundaries with the unknown loads. Lastly, further work is required in order to precisely estimate the absolute errors of the solutions. This definition will help to optimize the sensor locations, which is also a major issue when the available measurement information is limited like in structural monitoring.

REFERENCES

[1] Avril, S. and al., 2008, "Overview of Identification Methods of Mechanical Parameters Based on Full-field Measurements" , *Experimental Mechanics*, 48, pp.381-402.

[2] Hochard, Ch. "A Trefftz approach to computational mechanics", *International Journal for Numerical Methods in Engineering*, 56, pp. 2367-2386.

[3] Enzmann, M. and al. 1998, "Robust shape control of flexible structures using strain-measurement-gauges and piezoelectric stack actuator", *4th ESSM and 2nd MIMR Conference*.

[4] Tikhonov, A.N. and Arsenin, V.Y., 1977, "Solution of ill-posed problems", Winston.

[5] Nashed Z., 1987, A new approach to classification and regularization of ill-posed operator equations, *Inverse and ill-posed problems*, HW Eng and CW Groetsch

[6] Bogert, P.B. and al., 2003, "Structural shape identification from experimental strains using a modal transformation technique", *44th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Material Conference and Exhibit*, AIAA Inc.

[7] Tessler A., and Spangler J.L., 2005, "A least-squares variational method for full-field reconstruction of elastic deformations in shear-deformable plates and shells", *Computer Method in Applied Mechanics and Engineering*, 194, pp. 327-339.

[8] Salzmann M. and al., 2007, "Surface Deformation Models for Nonrigid 3D Shape Recovery", *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29, pp. 1481-1487.

[9] Hochard and al., 1993, "A simplified analysis of elastic structures", *European Journal of Mechanics A Solids*, 12(4), pp. 509-535

[10] Martini, D. and al., 2012 "Load identification for full-field reconstruction: applications to plates under tension loads", *International Journal of Numerical Method in Engineering*, in press.