A Bayesian Approach for Identification of Structural Crack Using Strain Measurements

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ABSTRACT

This paper proposes a statistical approach for identifying crack in structure using strain measurements and Bayesian inference, in which uncertainties from modeling error and measurement noise are explicitly included. The Bayesian approach is a model-based method, the crack is first represented by a set of parameters, i.e., coordinates of the two endpoints of the crack. An array of strain sensors is mounted on the structure to gather strain measurements under a known static loading. A forward model based on extended finite element method (XFEM) characterizing the strain responses of the structure with crack is incorporated in the identification procedure. By combining the measurement data and the prior information, Bayes' Theorem is used to update the probability distributions of the parameters of crack. A Markov chain Monte Carlo (MCMC) algorithm is employed for sampling the parameters' posterior distributions. Numerical study is conducted to demonstrate the effectiveness of the proposed method.

INTRODUCTION

The increasing emphasis on integrity of critical structures such as aircrafts, civil infrastructures, nuclear reactors, pressure vessels, etc., urges the needs to monitor structures in-situ and real-time to detect damages at an early stage to prevent catastrophic failure. With the advances in the area of smart materials and structures, it is possible to develop structural health monitoring (SHM) technologies that can be integrated into the structures as a built-in diagnosis system [1].

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Mathematically speaking, determination of the physical conditions of a structure based on the sensor signals is a nonlinear inverse problem. A lot of methods have been proposed to solve this kind of inverse problems, most of which are deterministic approaches [2]. In practice, uncertainties from modeling error, measurement noise, and other sources always cause troubles in solving inverse problems. Under such a circumstance, statistical approaches may be more appropriate than deterministic approaches in that probability distributions can be used to quantify the various uncertainties. In particular, the Bayesian statistical framework in which measurement data can be used to update the belief in the identification results, has previously been established and applied to structural systems by Beck and his colleagues [3-5]. One outstanding advantage of the Bayesian approach is that engineering judgments or expert knowledge can be easily incorporated into the analysis as prior information to reduce the uncertainties. Rather than pinpointing a single solution using deterministic approaches, the Bayesian approach can provide the probability density function (PDF) of the system parameters, giving both point and interval estimates [6-8].

The aim of this work is to provide a statistical Bayesian approach to identify crack in structure using strain measurements while considering uncertainties from modeling error and measurement noise. The paper is structured as follows. First, the Bayesian approach to crack identification is described. Then, a brief introduction of extended finite element (XFEM) and Markov chain Monte Carlo (MCMC) method and their application in this study is presented. In addition, numerical study is conducted to demonstrate the effectiveness of the proposed method. Finally, concluding remarks are given.

BAYESIAN APPROACH TO CRACK IDENTIFICATION

The chief idea of the Bayesian statistical identification framework is that it treats the system parameters, usually denoted by a vector $\boldsymbol{\theta}$, as random variables with joint distribution $p(\boldsymbol{\theta})$. In contrast to the deterministic identification approaches, this statistical approach aims to calculate the posterior (updated) distributions of the uncertain system parameters for a given set of measurement data. The final parameter estimate can be taken as the mean value of the posterior or use the value that maximizes the posterior distribution.

Consider a structure with one straight crack. The crack can be represented by the parameter vector as $\boldsymbol{\theta} = [x_1, y_1, x_2, y_2]^T$, in which (x_1, y_1) and (x_2, y_2) are the coordinates of the crack's two endpoints. An array of strain sensors are deployed on the structure to measure the strain responses under a given static loading to provide measurement data $\mathbf{D} = \{\varepsilon_i^m(x_i, y_i), i = 1, 2, \ldots, N_o\}$, in which $\varepsilon_i^m(x_i, y_i)$ is the measured strain at location (x_i, y_i), N_o is the number of sensors. Assume the modeling uncertainty and measurement uncertainty are both Gaussian type, the probabilistic description of the measured strain at a specific sensor location can be expressed as

$$\varepsilon_i^m = \varepsilon_i^c(\boldsymbol{\theta}, M_f) + e_i \tag{1}$$

where ε_i^c is the noise-free strain response calculated from a forward model M_f describing the static characters of a cracked structure subject to a known static loading and crack parameter $\boldsymbol{\theta}$, e_i is a Gaussian error. In this case, the likelihood function can be written as

$$p(\mathbf{D}\,|\,\boldsymbol{\theta},\sigma_\varepsilon^2)=\frac{1}{(2\pi\sigma_\varepsilon^2)^{N_o/2}}\exp\left[-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{N_o}(\varepsilon_i^m-\varepsilon_i^c(\boldsymbol{\theta},M_f))^2\right] \qquad (2)$$

where σ_ε^2 is the variance and assumption is made that the uncertainties from different sensors have the same variance across all sensors and they are uncorrelated. The likelihood function $p(\mathbf{D}\,|\,\boldsymbol{\theta},\sigma_\varepsilon^2)$ is a probabilistic statement about the distribution of the measurement data $\mathbf{D}$ given a forward model M_f and crack parameter $\boldsymbol{\theta}$. In this study, the forward model M_f is given before crack identification, and model selection is not considered. For convenience, the sum of squares in the likelihood in equations (2) can be defined as

$$Q(\mathbf{D},\boldsymbol{\theta})=\sum_{i=1}^{N_o}(\varepsilon_i^m-\varepsilon_i^c(\boldsymbol{\theta},M_f))^2 \qquad (3)$$

Using Bayes' Theorem, the posterior PDF of the parameters for a given set of measurement data is constructed by relating the prior PDF and the likelihood function as

$$p(\boldsymbol{\theta},\sigma_\varepsilon^2\,|\,\mathbf{D})=\frac{p(\mathbf{D}\,|\,\boldsymbol{\theta},\sigma_\varepsilon^2)p_\pi(\boldsymbol{\theta},\sigma_\varepsilon^2)}{p(\mathbf{D})} \qquad (4)$$

where $p(\boldsymbol{\theta},\sigma_\varepsilon^2\,|\,\mathbf{D})$ is the joint posterior distribution of $\boldsymbol{\theta}$ and σ_ε^2, $p_\pi(\boldsymbol{\theta},\sigma_\varepsilon^2)$ is the joint prior distribution of $\boldsymbol{\theta}$ and σ_ε^2, and

$$p(\mathbf{D})=\int p(\mathbf{D}\,|\,\boldsymbol{\theta},\sigma_\varepsilon^2)p_\pi(\boldsymbol{\theta},\sigma_\varepsilon^2)d\boldsymbol{\theta}d\sigma_\varepsilon^2 \qquad (5)$$

is a normalizing constant that ensure the integration of the PDF over the predefined domain is equal to unity. For each of the parameter θ_j in $\boldsymbol{\theta}$, the marginal posterior distribution can be obtained by integrating equation (5) with respect to the rest parameters and variance σ_ε^2 over the domain of interest as

$$p(\theta_j\,|\,\mathbf{D})=\int p(\boldsymbol{\theta},\sigma_\varepsilon^2\,|\,\mathbf{D})d\boldsymbol{\theta}_{-j}d\sigma_\varepsilon^2\propto\int p(\mathbf{D}\,|\,\boldsymbol{\theta},\sigma_\varepsilon^2)p_\pi(\boldsymbol{\theta})d\boldsymbol{\theta}_{-j}d\sigma_\varepsilon^2 \qquad (6)$$

where the notation $\int d\boldsymbol{\theta}_{-j}d\sigma_\varepsilon^2$ denotes the multidimensional integral over σ_ε^2 and all the rest parameters other than θ_j in $\boldsymbol{\theta}$.

Equation (6) gives a general expression of the updated marginal PDF for each parameter of the crack using measured strain responses. However, evaluation of equation (6) is usually difficult since it involves integration over multi-dimensional parameters. In this study, MCMC method is employed for drawing the posterior distributions for the parameters of the crack.

XFEM FOR CRACK MODELLING

Originally, the XFEM algorithm was developed to enable the modelling of crack growth without remeshing [9,10]. In order to incorporate stress and displacement fields which are discontinuous across the crack, the mesh in traditional formulations of the finite element method had to be adapted so that the crack coincided with the element edges. In contrast, the XFEM algorithm allows for the crack to pass arbitrarily though elements by incorporating enrichment functions to handle the field discontinuities. In this manner, the mesh can remain fixed throughout the evolution of the crack. In this study, as MCMC is employed to solve the Bayesian updating problem, the location and size of the crack need to be iteratively updated. Using XFEM to model the crack could avoid remeshing the domain during the identification iterations [11].

The key idea of XFEM is to locally enrich the standard finite element approximation with local partitions of unity enrichment functions which are chosen according to the physics of the problem.

The XFEM approximation $\mathbf{u}^h$ of a cracked domain takes the form as [9]

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i \in I} \mathbf{u}_i N_i(\mathbf{x}) + \sum_{j \in J} \mathbf{b}_j N_j(\mathbf{x}) H(\mathbf{x}) + \sum_{k \in K} N_k(\mathbf{x}) \left(\sum_{l=1}^{4} \mathbf{a}_k^l B_l(r,\theta) \right) \tag{7}$$

where $N_i(\mathbf{x})$ is the shape function associated with node i, J is the set of all nodes whose support is bisected by the crack and K contains all the nodes of the elements containing the crack tip as illustrated in figure 1. The nodal degree of freedom corresponding to the displacement are $\mathbf{u}_i$, $\mathbf{b}_j$ and $\mathbf{a}_k$.

The important and distinguishing factor to note in equation is the enrichment functions $H(\mathbf{x})$ and $B_l(r,\theta)$. The Heaviside function $H(y)$ is defined as

$$H(y) = \begin{cases} 1 & y > 0 \\ -1 & y < 0 \end{cases} \tag{8}$$

This implies that the discontinuity occurs at the location of the crack. The branch function B_l is defined by

$$B_l(r(\mathbf{x}), \theta(\mathbf{x})) = \left\{ \sqrt{r}\sin\frac{\theta}{2} \quad \sqrt{r}\cos\frac{\theta}{2} \quad \sqrt{r}\sin\frac{\theta}{2}\sin\theta \quad \sqrt{r}\cos\frac{\theta}{2}\sin\theta \right\} \tag{9}$$

where (r,θ) is a polar coordinate system with its origin at the crack tip and $\theta = 0$ tangent to the crack at its tip. The above functions span the asymptotic crack tip solution of elasto-statics, and $\sqrt{r}\sin(\theta/2)$ takes into account the discontinuity across the crack face.

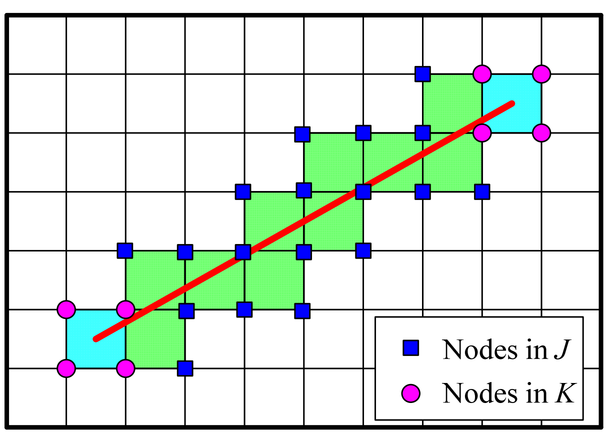

Figure 1. Sets of nodes selected for enrichment.

MARKOV CHAIN MONTE CARLO METHOD

Posterior distributions used in Bayesian inference are often complicated, making it difficult to draw independent samples for standard Monte Carlo method. Under such a situation, MCMC simulation is usually employed as an alternative choice for sampling. The result of MCMC is a dependent sequence of samples (a Markov chain) that has stationary distributions equal to the target distribution.

In this study, a MCMC algorithm for Gaussian likelihood and uniform transition distribution developed by Nichols et al. [12] is employed to sample the posterior distributions of the crack parameters $\boldsymbol{\theta}$. Compared to standard Metropolis-Hastings algorithm, this algorithm combines Gibbs sampling concept that the full conditional distribution of each parameter can be thought of as its posterior distribution if other parameters' values are known. It updates each parameter sequentially by using the most recent sampled values. Also, in this algorithm, for each parameter, the interval $2L$ has been continuously tuned during the "burn-in" period to achieve an appropriate acceptance rate and improve the performance of MCMC. In addition, it assumes a diffuse gamma prior on the precision parameter $1/\sigma_\varepsilon^2$ which is the case of conditional conjugacy, thus the variance σ_ε^2 can be directly sampled from an inverse gamma distribution. More detailed information about this algorithm can be referred to reference [12].

The proposed MCMC algorithm for generating the posterior parameter distributions $p(\theta_j)$ for $\boldsymbol{\theta} = [x_1, y_1, x_2, y_2]^T$ given the forward XFEM model M_f, the measurement data $\mathbf{D}$, and prior parameter distributions $p_\pi(\theta_j)$ are described in following steps:

Step 1 Set number of total iterations N_T and number of "burn-in" iterations N_B.

Step 2 Initialize the chain with iteration number $i = 0$, initial guesses for parameter values (randomly chosen from the priors) $\boldsymbol{\theta}(0) = \theta_j(0)$ and initial values for tuning parameters L_j, initial variance sampled from inverse gamma distribution $\sigma_\varepsilon^2(0) = IG(N_o/2+1, Q(\mathbf{D}, \boldsymbol{\theta}(0))/2)$, where $IG(\alpha, \beta)$ is the inverse gamma distribution with parameters α and β, and $Q(\mathbf{D}, \boldsymbol{\theta})$ is the sum of squares in the likelihood defined in equation (5).

Step 3 Increase i by 1 and for each parameter θ_j generate a candidate

$$\theta_j^* = \theta_j(i-1) + 2L_j \times U(-1,1),$$

compute $r = \dfrac{p_\pi(\boldsymbol{\theta}^*)}{p_\pi(\boldsymbol{\theta}(i-1))} \exp(-\dfrac{\sigma_\varepsilon^2(i-1)}{2} \times (Q(\mathbf{D}, \boldsymbol{\theta}^*) - Q(\mathbf{D}, \boldsymbol{\theta}(i-1))))$

where

$\boldsymbol{\theta}^* \equiv (\theta_1(i), \cdots, \theta_{j-1}(i), \theta_j^*, \theta_{j+1}(i-1), \cdots)$ and

$\boldsymbol{\theta}(i-1) \equiv (\theta_1(i), \cdots, \theta_{j-1}(i), \theta_j(i-1), \theta_{j+1}(i-1), \cdots)$.

Step 4 Randomly generate a number R from the uniform distribution $U(0,1)$, if $R < r$, set $\theta_j(i) = \theta_j^*$, and adjust tuning parameter $L_j = L_j \times 1.01$, else reject the new value, keep $\theta_j(i) = \theta_j(i-1)$, and adjust tuning parameter

$L_j = L_j / 1.07$, directly sample the variance

$\sigma_\varepsilon^2(i) = IG(N_o / 2 + 1, Q(\mathbf{D}, \boldsymbol{\theta}(i)) / 2)$.

Step 5 After $i > N_B$ iterations, cease adjusting the tuning parameters L_j and run the same procedure as Steps 3 and 4, record subsequent values $\theta_j(i)$ as members of stationary Markov chains that can represent the posterior distributions $p(\theta_j)$.

NUMERICAL EXAMPLE

To illustrate the effectiveness of the proposed method, numerical example for a square plate is studied here. The dimensions of the plate is 1×1 (units), and uniform distributed tension are applied on both the left and right vertical edges of the plate which is under MODE I tension along the horizontal direction as illustrated in figure 2. The numerical data came from the forward XFEM model with finer mesh due to lack of actual experimental data. The sensor locations are labeled as '⊕' in figure 2. For considering the measurement error, zero-mean Gaussian white noises are added to the numerical data as $\varepsilon_i^m = \varepsilon_i^{XFEM}(1+\eta)$, where ε_i^{XFEM} are the strain data calculated from the forward XFEM model, and η is a Gaussian variable with zero mean and standard deviation *s* [8].

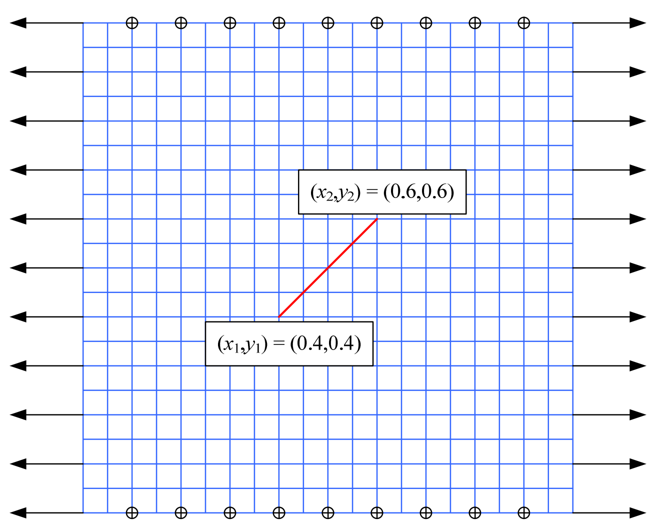

Figure 2. Mesh generation, crack orientation and sensor placement in numerical example.

With simulated measurement data and assumed uniform priors, the proposed statistical identification approach is implemented to obtain the posterior distributions of the parameters about the crack. Figure 3 shows the samples of crack parameters for the first and second endpoints by MCMC method when $\eta = 2\%$. For each parameter, totally 2000 samples are obtained by MCMC in which the first 500 are set as the "burn-in" period. Figure 4 and figure 5 show the histograms formed by the rest 1500 samples for each parameter from figure 3. Then estimated PDF of distributions for coordinates of crack endpoints can be obtained from the histograms, and each parameter can be estimated using the values maximizing the PDF.

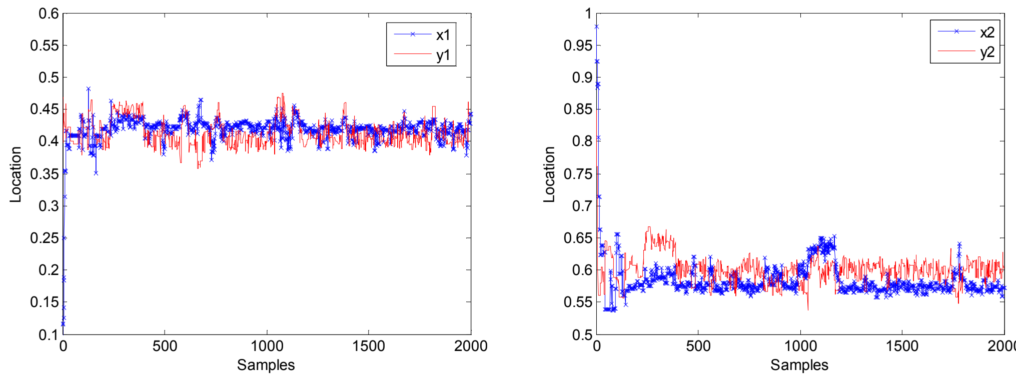

Figure 3. Samples of the coordinates of first endpoint (left) and second endpoint (right) by MCMC.

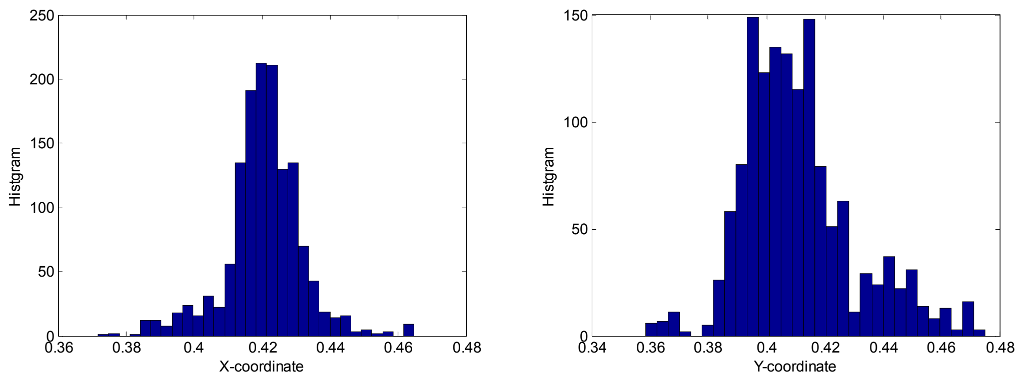

Figure 4. Histograms of identified x-coordinate (left) and y-coordinate (right) for first endpoint.

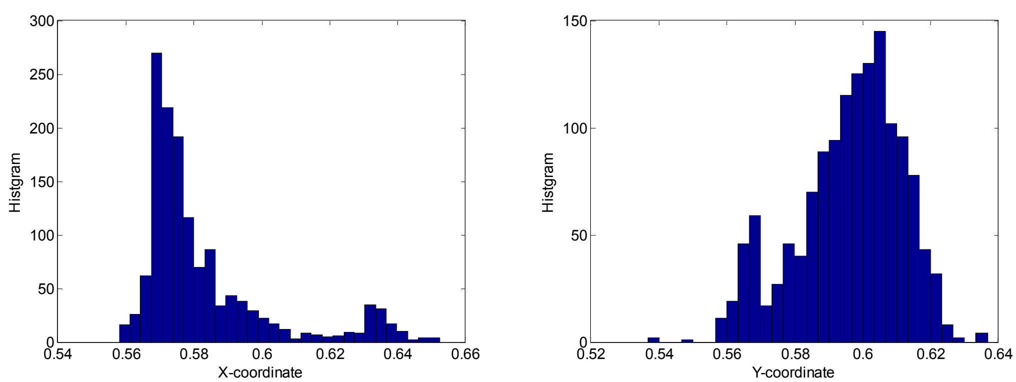

Figure 5. Histograms of identified x-coordinate (left) and y-coordinate (right) for second endpoint.

CONCLUSIONS

A statistical Bayesian approach for identifying crack in structure is proposed in this study. A forward model based on XFEM characterizing the strain responses of a structure with crack is incorporated in the identification procedure. By combining the measurement data and the prior information, Bayes' Theorem is used to update the probability distributions of the parameters of crack. A MCMC algorithm is employed for sampling the parameters' posterior distributions. The effectiveness of the proposed approach is validated and demonstrated by numerical study. The results have shown that, by using MCMC sampling method, the Bayesian approach successfully identified the crack parameters for the nonlinear inverse problem in which uncertainties from modeling error and measurement noise are explicitly considered. Rather than pinpointing a single solution using deterministic approaches, the Bayesian approach provides the probability distribution of the crack parameters, giving a possible uncertainty analysis to the estimates.

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