A Generalized Equivalent Loading Model for Piezoelectric Elements

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ABSTRACT

Typically, an active structural health monitoring system (SHM) consists of an integrated network of actuator/sensor piezoceramic elements that inject and receive ultrasound pulses into the host structure collecting information about structural health. Although numerical simulation has been used extensively in understanding and aiding in the design of such systems, analytical models are still the primary vehicle for understanding actuation and sensing mechanisms. Being based on simplified assumptions it suffers from certain limitations with respect to its extension to SHM systems. The main assumption of equivalent loading is neglecting the mechanical coupling by replacing the piezoceramic with equivalent load(s). The present work addresses two of the limitations of this assumption, namely the effect of the thickness of the piezoceramic element on the equivalent load in dynamic setting at high frequency. This study is done via a novel formulation based on earlier work considering Lamb waves as a propagating carrier wave with superimposed modes which is not limited to isotropic media and the inclusion of the generalized loads are done via the reciprocity relation. The model results are compared with the numerical simulation results using commercial finite element software (ANSYS) for a wide range of frequencies. The applicability of the model to frequencies as large as 0,5 MHz is demonstrated. The effect of the loading on the energy partitioning between Lamb fundamental modes, without the need for prior adjustment comes as the first advantage of the presented model over the classical integral transform based models, thus enabling a direct relation for mode tuning. The second advantage is the easy inclusion of the finite frequency content of the excitation through Fourier transform, relaxing the assumption of harmonic waves propagation that prevails in the classical models enabling a more realistic signal to be modeled.

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INTRODUCTION

The representation of the displacement fields as summations over normal modes of vibration or wave propagation of an elastic domain is a common method in solving excitation problems. However, for the dynamic response of an infinite elastic plate involving Lamb modes, this procedure has not been applied extensively until relatively recently. The modes of wave propagation in an elastic plate are well known since Lamb's classical work, the Rayleigh-Lamb frequency equation is well understood and comprehensively analysed, yet till the work of Achenbach and Xu [1,2,3], no direct way in using these modes to deduce the displacement fields has been implemented. Achenbach and Xu work provided a usable orthogonality relation and a suitable method to obtain the modal coefficients. Although these relations have been published in the literature for nearly 15 years now, it did not find its way yet to the SHM field. The main objective of this paper is to test the applicability of this novel formulation to the practical applications related to the structural health monitoring field. A review of the novel formulation of Lamb waves is presented in the following section, followed by the use of reciprocity relation to formulate the orthogonality condition to relate the kinematic modal description to the general mechanical excitation. The second section will proceed to formulate the piezoelectric excitation as equivalent mechanical loading, and then formulating the displacement field induced by such an excitation in terms of summation over Lamb modes in Achenbach's formulation. This leads to explicit relation for the frequency mode tuning.

MODAL FORMULATION FOR LAMB WAVES

Following Achenbach [1], the Cartesian components shown in Figure 1 of the displacement field are expressed as:

$$u_1(\mathbf{x},t) = \frac{1}{k}\, V(x_3)\frac{\partial\phi(x_1,x_2)}{\partial x_1}\, e^{-i\omega t} \tag{1}$$

$$u_2(\mathbf{x},t) = \frac{1}{k}\, V(x_3)\frac{\partial\phi(x_1,x_2)}{\partial x_2}\, e^{-i\omega t} \tag{2}$$

$$u_3(\mathbf{x},t) = W(x_3)\,\phi(x_1,x_2)e^{-i\omega t} \tag{3}$$

where k is the wavenumber. Equations (1)-(3) satisfy the elastodynamic equations of motion if the dimensionless function $\phi(x_1,x_2)$ is a solution of the reduced membrane equation:

$$\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + k^2\phi = 0$$

and $V(x_3)$, and $W(x_3)$ are solutions of the following ODEs system:

$$(\lambda + \mu)W' + \frac{\mu}{k}\, V'' + \frac{\rho\omega^2}{k}\, V = k\,(\lambda + 2\mu)V \tag{4}$$

$$(\lambda + 2\mu)\, W'' + \rho\omega^2\, W = k\,(\lambda + \mu)V' + k^2\mu W \tag{5}$$

where λ and μ are Lamé's constants, and ρ is the mass density. Solutions of form (1)-(3) are particularly convenient for Lamb waves; the plate mid-plane is parallel to

$x_1 - x_2$ plane. In such a formulation ϕ represents a carrier wave for the propagation in the plate plane, while V, and W describe the thickness motions for Lamb waves and the associated thickness dependence. The enforcement of the traction free boundary conditions at the parallel surfaces ($x_3 = \pm h$) of the plate yields the Rayleigh-Lamb frequency equation, a noticeable fact is that no plane strain assumption was used, making this formulation a general one for plate waves.

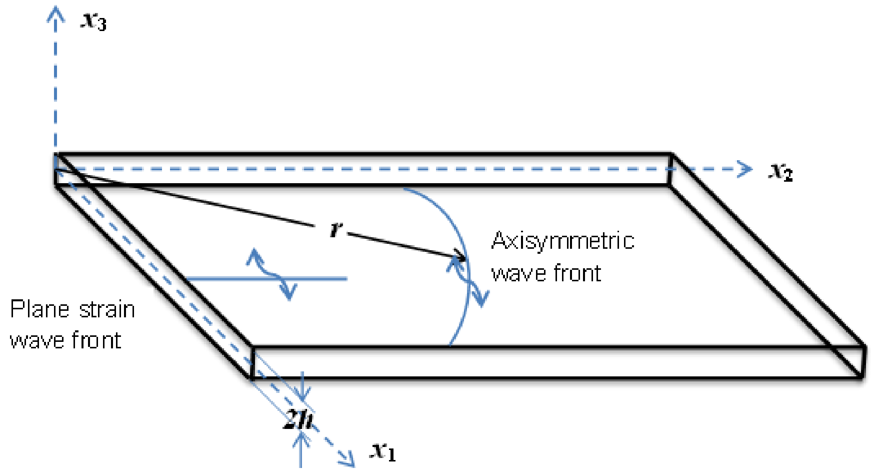

Figure 1: The Cartesian coordinate system used in the formulation.

For Lamb modes at a given frequency ω, the wavenumber k_n is the solution of the Rayleigh-Lamb frequency equation associated with the n^{th} Lamb mode as shown in Fig. 2, which, according to their symmetry with respect the plate mid-plane, are further divided into symmetric S_n and antisymmetric A_n modes. For symmetric modes, where the wavenumber is referred to as k_{Sn}, the superimposed thickness motion takes the form:

$$V_n^S(x_3) = A_n^S(s_1 \cos(p\, x_3) + s_2 \cos(q\, x_3)) \tag{6}$$
$$W_n^S(x_3) = A_n^s(s_3 \sin(p\, x_3) + s_4 \sin(q\, x_3)) \tag{7}$$

and for the antisymmetric n^{th} mode with wavenumber k_{An}:

$$V_n^A(x_3) = A_n^A(a_1 \sin(p\, x_3) + a_2 \sin(q\, x_3)) \tag{8}$$
$$W_n^A(x_3) = A_n^A(a_3 \cos(p\, x_3) + a_4 \cos(q\, x_3)) \tag{9}$$

where A_n^S and A_n^A are the modal amplitudes that will further be determined based on the excitation and the constants $s_{1\ldots4}$ and $a_{1\ldots4}$ are given in [1].

For axisymmetric wave propagation, the carrier wave for an outgoing wave is expressed using the Hankel function:

$$\phi(r) = H_0^2(r), \qquad \text{with } r = \sqrt{(x_1^2 + x_2^2)}$$

and for the plane strain case, cf. Fig 1:

$$\phi(r) = e^{-ik_n x_1}$$

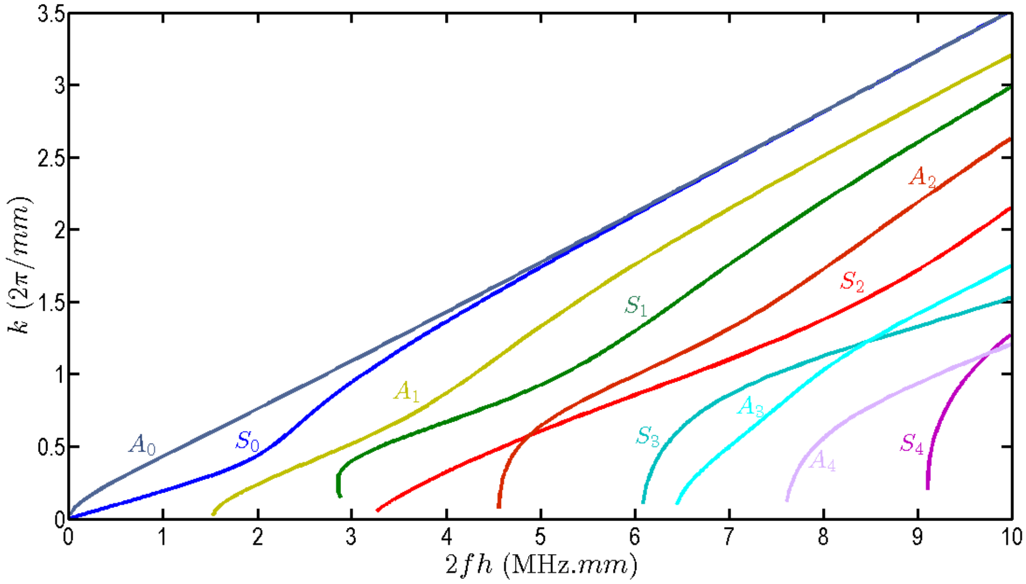

Figure 2: Dispersion Curves for aluminum plate of thickness 2h = 1 mm.

Equations (2) and (3) can then be simplified and can be rewritten as:

$$u_2^{Sn[An]} = V_n^{S[A]}(x_3)\ i\ A_n^{S[A]}\ e^{-i(\omega t + k_n x_1)} \tag{10}$$

$$u_3^{Sn[An]} = W_n^{S[A]}(x_3)\ A_n^{S[A]}\ e^{-i(\omega t + k_n x_1)} \tag{11}$$

This formulation provides a kinematic description for the admissible motions under the traction free boundary conditions. The main noticeable aspect of this formulation is that it does not assume a priori plane strain condition, thus extending its applicability to anisotropic media and plane stress conditions.

Reciprocity theorems in elasticity theory provide a relation between displacements, traction components and body forces for two different loading states of a single body or two bodies of the same geometry [4, 5]. Through the use of reciprocity and dummy wave solution, Achenbach and Xu [1] formulated a modal orthogonality rule that could be used to couple the load with the modal amplitude. For a horizontal load *Q*, the modal coefficients for the fundamental Lamb modes are:

$$A_n^S = \frac{k_n}{4i}\frac{QV_n^S(h)}{I_{nn}^S} \tag{12}$$

$$A_n^A = \frac{k_n}{4i}\frac{QV_n^A(h)}{I_{nn}^A} \tag{13}$$

where $I_{nn}^{S[A]}$ is the orthogonality relations derived via the reciprocity theorem, as given by equations (96)-(106) in [1].

The expressions (12) and (13) provide a direct relation for the modal amplitudes, and their dependence on the frequency. From engineering point of view, an advantage is the ability to separate the effects of each variable directly.

PIEZOCERAMIC EXCITATION

When the piezoceramic elements are bonded to the surface of the plate, and subjected to a time varying voltage, it expands and contracts therefore generating shear stress at the interface between the plate and the actuator. In the case of perfect bonding the shear stresses accumulate at the periphery of the actuator. This model assumes that the actuator and the plate are two separate bodies and that the only manifestation of the actuator is in the shear traction created at the idealized interface. This model is consistent with the perfect bonding assumption, i.e. the shear stress is concentrated in a small area close to the edge of the actuator, leading to the well-known pin force model that originally developed in the active structures field [6]. For this model, equations (10)-(11) give for a piezoceramic of length l:

$$u_2^{Sn[An]} = V_n^{S[A]}(x_3)\, i\, A_n^{S[A]} \left(1 + e^{ik_n l}\right) e^{-i(\omega t + k_n x_1)} \tag{14}$$
$$u_3^{Sn[An]} = W_n^{S[A]}(x_3)\, A_n^{S[A]} \left(1 + e^{ik_n l}\right) e^{-i(\omega t + k_n x_1)} \tag{15}$$

This gives a sinusoidal dependence on the wavelength, modulated by the modal amplitude, which is dependent only on the properties of the media and loading conditions. The equivalent pin forces for a piezoceramic ideally bonded to the surface of the plate are divided into symmetric and antisymmetric loading as shown in Fig 3.

The effect of the piezoceramic thickness, and its mechanical coupling with the plate is not taken into account in the pin-force model, the actuator is assumed to remain plane, this assumption ignores the bending behavior of the piezoceramic being constrained by the plate. The reciprocity formulation, enables the inclusion of this moments easily with minor modifications to the value of the load entering the antisymmetric mode coefficient A_n^A.

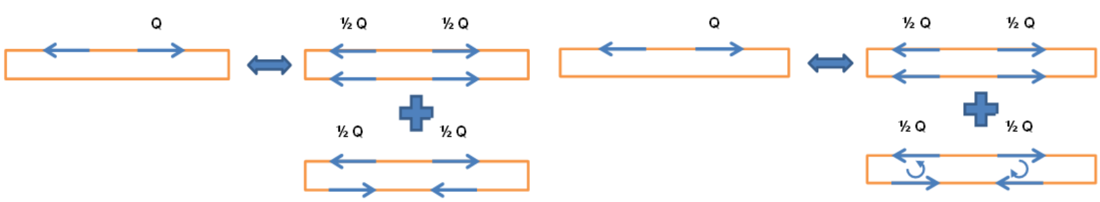

Figure 3: Division of pin force loading to symmetric and antisymmetric loading (left), and the effect of the piezoceramic thickness represented by the added moment (right).

The forces and moments induced by the actuator could be expressed as [7]:

$$Q = -\frac{E_{pzt} h_{pzt} d_{31}}{1 - \nu_{pzt}} V_{pzt} \tag{16}$$

$$M = -\left(\frac{1}{8}\right) \frac{E_{pzt}}{1 - \nu_{pzt}} \left[4 \left(\frac{h}{4} + h_{pzt}\right)^2 - \frac{h^2}{4} \right] \frac{d_{31}}{h_{pzt}} V_{pzt} \tag{17}$$

where $E_{pzt}, \nu_{pzt}, d_{31}, h_{pzt}$ and V_{pzt} are respectively the Young's modulus, Poisson's ratio, piezoelectric coupling coefficient, thickness and applied voltage of the piezoceramic element.

This modifies the equation for the antisymmetric mode coefficient only by replacing Q in equation (13) by $Q + Mh$ which is manifested in an increase of the amplitude of the antisymmetric mode with respect to the symmetric mode.

Figure 4 shows a comparison between the predicted amplitudes of the in-plane displacement of the fundamental Lamb modes with the original pin-force model (without moment) and the modified pin-force model (with moment).

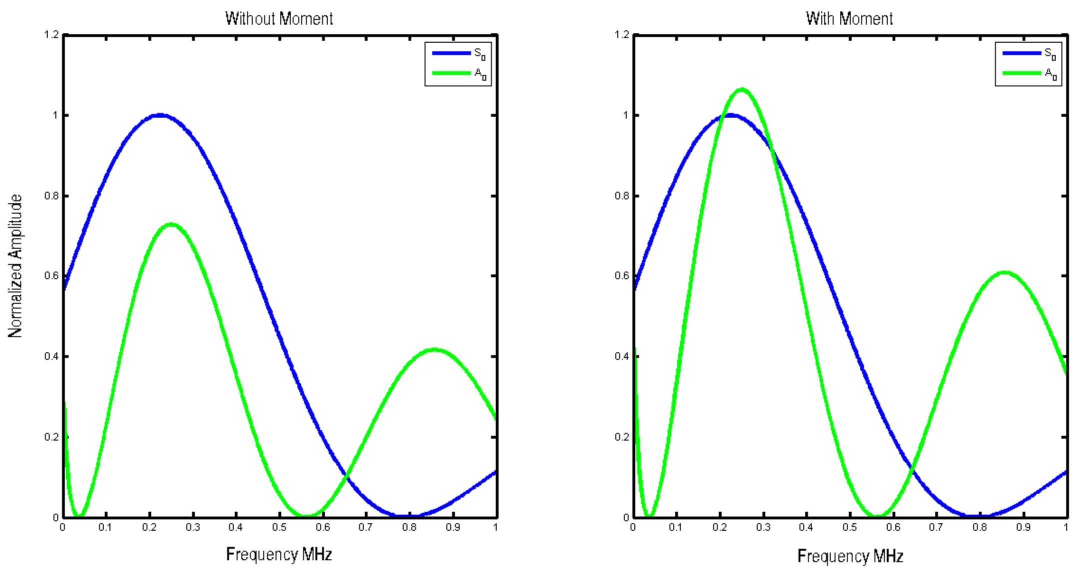

Figure 4: The predicted in-plane displacement amplitude without including the actuator bending effect (left) and with the bending effect (right).

For a time dependent loading with a finite frequency content, $Q(t)$, the function can be represented by a Fourier integral:

$$Q(t) = \int_{-\infty}^{\infty} \hat{Q}(\omega) e^{-i\omega t}\, d\omega \qquad , \qquad \hat{Q}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(t) e^{-i\omega t}\, dt$$

leading to a direct relation for the dependence of the amplitudes on the frequency content of a finite signal.

THEORETICAL AND NUMERICAL RESULTS

Figure 5 shows the results of the in-plane displacement as a function of the frequency of excitation with the inclusion of the moment, compared with the FEM results of ANSYS simulation of a piezoceramic element bonded to the surface of an aluminum plate, in the time domain, at a point 200 mm away from the center of the actuator, a plane strain case was assumed, and the time dependent voltage was a sinusoidally modulated 6.5 cycles, with 10 V amplitude, the piezocermaic element was 5 mm in diameter and 0.25 mm thickness. In order to extract the maximum amplitude of fundamental modes from the time trace signal, Hilbert transform was applied to the signal, and the maximum of the real part was extracted, for each mode. The properties of the aluminum plate and the piezoceramic element used in the simulation are listed in Table 1.

Table 1: The properties of aluminum plate and PZT.

	Aluminum	Piezoceramic
Young's Modulus GPa	67	106
Poisson's ratio	0.33	0.35
Density kg/m^3	2700	7650
d_{31} (C/N)	------	175×10^{-12}

The aluminum plate was made long enough to avoid reflection form the boundaries, the length is dependent on the frequency content and the fastest mode of the two fundamental modes $L = \max\,(C_{g(S0)})\,\mathrm{t_{final}}$. The maximum group velocity corresponds to the maximum frequency with considerable power in the excitation signal. The thickness of the aluminum plate is 1.54 mm.

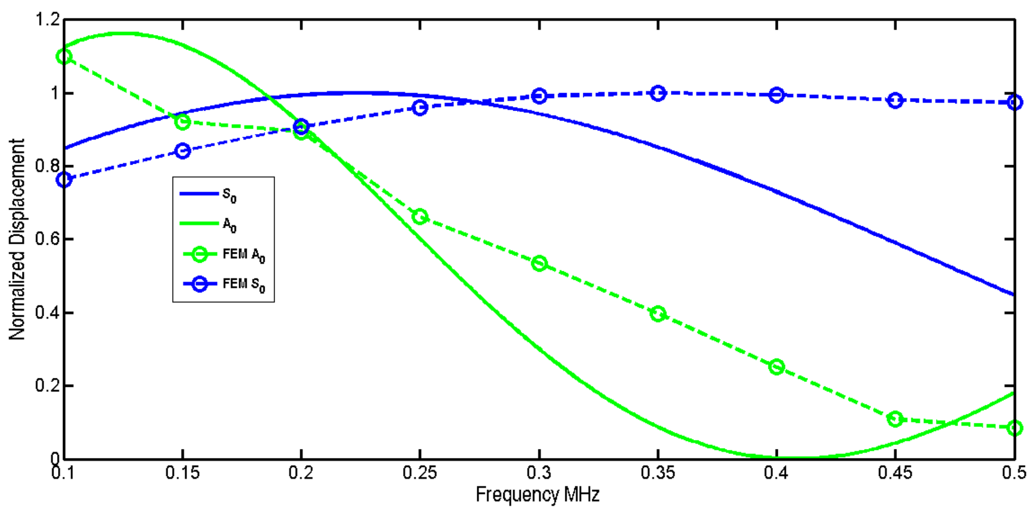

Figure 5: The predicted in-plane displacement amplitude without including the actuator bending effect (left) and with the bending effect (right).

The comparison between the FEM results and the theoretical results shows a considerable shift in the modes minima and maxima toward higher frequency for the PZT coupled simulation as opposed to the theoretical curves. The FEM is a two dimensional model, which excludes the possibility of the change in the actuator mode.

CONCLUSIONS

This preliminary study explored a recently new formulation based on the elastodynamic reciprocity coupled with a dummy wave solution to extract a usable orthogonality relation that enables the direct determination of the modal amplitudes associated with the Lamb modes. The applicability of this model to the piezoceramic excitation for SHM purposes was explored. The simplicity of the model and the ability of effects separation come as the first advantages with respect to the SHM preliminary concept development. The model is more general and its applicability could be extended to more applications.

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